

Associative Submanifolds of the 7-Sphere

Jason D. Lotay

Imperial College London

Abstract

We perform the first major study of associative submanifolds in the 7-sphere \mathcal{S}^7 , which are minimal 3-dimensional submanifolds associated with the natural G_2 structure on \mathcal{S}^7 . We classify the associative 3-folds which arise as group orbits and those with constant curvature. We also describe the associative 3-folds satisfying the curvature condition known as Chen's equality using the theory of ruled associative 3-folds. Finally, we produce \mathcal{S}^1 -families of non-congruent isometric associative submanifolds in \mathcal{S}^7 satisfying Chen's equality from general minimal 2-spheres in \mathcal{S}^6 .

1 Introduction

Associative submanifolds of the 7-sphere \mathcal{S}^7 are 3-dimensional *minimal* submanifolds which are related to the G_2 structure that \mathcal{S}^7 naturally inherits from the standard $\text{Spin}(7)$ structure on \mathbb{R}^8 . They were introduced by Harvey and Lawson in [25] in their seminal work on *calibrated geometries*.

Examples of associative 3-folds in \mathcal{S}^7 are provided by well-known classes of submanifolds, including *Lagrangian* submanifolds of the nearly Kähler 6-sphere, Hopf lifts of *holomorphic curves* in \mathbb{CP}^3 and *minimal Legendrian* submanifolds of \mathcal{S}^7 . Associative submanifolds of \mathcal{S}^7 are the links of *Cayley* cones and are thus important in the study of singularities of Cayley submanifolds, which are calibrated 4-dimensional submanifolds of $\text{Spin}(7)$ manifolds. Understanding singularities of Cayley 4-folds will be essential in any proposed construction of a Cayley fibration of a compact $\text{Spin}(7)$ manifold.

In this article we perform the first major study of associative submanifolds of the 7-sphere. We begin in §2 by discussing G_2 structures and, specifically, the G_2 structure on \mathcal{S}^7 . This allows us, in §3, to define associative submanifolds in \mathcal{S}^7 . We describe the basic properties of these submanifolds and discuss their connection with other geometries. In §4 we derive the structure equations for

associative submanifolds in \mathcal{S}^7 , which will be invaluable for later sections.

Our first major results, proved in §5, classify the associative 3-folds in \mathcal{S}^7 which arise as orbits of closed 3-dimensional Lie subgroups of $\text{Spin}(7)$ as follows.

Theorem 1.1 *Let A be a connected associative 3-fold in \mathcal{S}^7 which is the orbit of a closed 3-dimensional Lie subgroup of $\text{Spin}(7)$. Suppose further that A does not lie in a totally geodesic \mathcal{S}^6 . Then, up to rigid motion, A is either*

- (a) $A_1 \cong \text{U}(1)^3$ given by Example 5.4,
- (b) $A_2 \cong \text{SU}(2)/\mathbb{Z}_3$ given by Example 5.9, or
- (c) $A_3 \cong \text{SU}(2)$ given by Example 5.10.

Note The example A_3 is novel and does not arise from other known geometries.

Theorem 1.2 *Let A be a connected, non-totally geodesic, associative 3-fold in \mathcal{S}^7 which lies in a totally geodesic \mathcal{S}^6 . Suppose further that A is the orbit of a closed 3-dimensional Lie subgroup of G_2 . Then, up to rigid motion, A is either*

- (a) $L_1 \cong \text{Sp}(1)$ given by Example 5.12,
- (b) $L_2 \cong \text{SO}(3)$ given by Example 5.13, or
- (c) $L_3 \cong \text{SO}(3)/A_4$ or $L_4 \cong \text{SO}(3)/S_3$ given by Example 5.14.

Since the subgroup of $\text{Spin}(7)$ fixing a real direction in \mathbb{R}^8 is isomorphic to G_2 , if A is a homogeneous associative 3-fold in \mathcal{S}^7 lying in a totally geodesic \mathcal{S}^6 , then it arises as the orbit of a subgroup of G_2 .

Note Theorem 1.2 is essentially due to Mashimo [36].

In §6 we study various natural curvature conditions which allow us to prove rigidity results for associative submanifolds in \mathcal{S}^7 . In particular, we classify the associative 3-folds with *constant curvature*.

Theorem 1.3 *Let $A(\kappa)$ be a connected associative 3-fold in \mathcal{S}^7 with constant curvature κ . Then either $\kappa = 0$, $\kappa = \frac{1}{16}$ or $\kappa = 1$. Moreover, up to rigid motion, $A(1) = A_0$ given by Example 5.1, $A(0) = A_1$ given by Example 5.4, and $A(\frac{1}{16}) = L_3 \subseteq \mathcal{S}^6$ given by Example 5.14.*

So that we may study *Chen's equality*, which is a curvature condition for minimal submanifolds of spaces with constant curvature, in §7 we review and build upon the analysis of associative submanifolds which are *ruled* by oriented geodesic circles in \mathcal{S}^7 using techniques in [19]. We can summarise the key results we need in the following corollary of [19, Proposition 2.1 & Theorem 7.8].

Theorem 1.4 *Let A be an associative submanifold in \mathcal{S}^7 which is ruled by oriented geodesic circles. Then either*

- (a) *A is the Hopf lift of a holomorphic curve in \mathbb{CP}^3 or*
- (b) *A is constructed from a minimal surface Σ in \mathcal{S}^6 and a holomorphic curve Γ in a \mathbb{CP}^1 -bundle $\mathcal{X}(\Sigma)$ over Σ as in Example 7.6.*

In Theorem 1.4(b), the holomorphic curve Γ defines an immersion of Σ in the space \mathcal{C} of oriented geodesic circles in \mathcal{S}^7 as a *pseudoholomorphic curve* with respect to a $\text{Spin}(7)$ -invariant almost complex structure. This viewpoint allows us to identify a distinguished class of pseudoholomorphic curves in \mathcal{C} , which we call *linear curves*, and enables us to prove the following in §8.

Theorem 1.5 *Let A be an associative submanifold in \mathcal{S}^7 which satisfies Chen's equality. Then either*

- (a) *A is the Hopf lift of a holomorphic curve in \mathbb{CP}^3 or*
- (b) *A is constructed from a minimal surface Σ in \mathcal{S}^6 as in Example 7.6, where Σ is isotropic and the pseudoholomorphic lift of Σ in \mathcal{C} is linear.*

Finally, in §9, we consider the problem of finding Riemannian 3-manifolds which admit families of *isometric embeddings* as associative submanifolds in \mathcal{S}^7 . Using the well-developed theory of minimal 2-spheres in \mathcal{S}^6 , we prove the following interesting result.

Theorem 1.6 *Given a non-constant curvature minimal \mathcal{S}^2 in \mathcal{S}^6 , there exists a Riemannian 3-manifold (A, g_A) , which is an \mathcal{S}^1 -bundle over \mathcal{S}^2 , and an \mathcal{S}^1 -family of non-congruent isometric associative embeddings of (A, g_A) in \mathcal{S}^7 which satisfy Chen's equality.*

Since there are a large number of minimal 2-spheres in \mathcal{S}^6 , this provides many examples of families of isometric associative submanifolds of \mathcal{S}^7 . Moreover, there is a Weierstrass representation for such minimal 2-spheres by [2], [10], [11], and thus for the resulting isometric associative submanifolds.

2 The G_2 structure on \mathcal{S}^7

The key to defining associative submanifolds of the 7-sphere is to introduce a G_2 structure on \mathcal{S}^7 , which is induced by the standard $\text{Spin}(7)$ structure on \mathbb{R}^8 . We begin by defining distinguished differential forms on \mathbb{R}^7 and \mathbb{R}^8 .

Definition 2.1 Let \mathbb{R}^7 have coordinates (x_1, \dots, x_7) and let \mathbb{R}^8 have coordinates (x_0, \dots, x_7) . For convenience we denote the form $dx_i \wedge dx_j \wedge \dots \wedge dx_k$ by $\mathbf{dx}_{ij\dots k}$. Define a 3-form φ_0 on \mathbb{R}^7 by:

$$\varphi_0 = \mathbf{dx}_{123} + \mathbf{dx}_{145} + \mathbf{dx}_{167} + \mathbf{dx}_{246} - \mathbf{dx}_{257} - \mathbf{dx}_{347} - \mathbf{dx}_{356}.$$

Note that the Hodge dual of φ_0 is:

$$*\varphi_0 = \mathbf{dx}_{4567} + \mathbf{dx}_{2367} + \mathbf{dx}_{2345} + \mathbf{dx}_{1357} - \mathbf{dx}_{1346} - \mathbf{dx}_{1256} - \mathbf{dx}_{1247}.$$

Define a 4-form Φ_0 on \mathbb{R}^8 by:

$$\begin{aligned} \Phi_0 = & \mathbf{dx}_{0123} + \mathbf{dx}_{0145} + \mathbf{dx}_{0167} + \mathbf{dx}_{0246} - \mathbf{dx}_{0257} - \mathbf{dx}_{0347} - \mathbf{dx}_{0356} \\ & + \mathbf{dx}_{4567} + \mathbf{dx}_{2367} + \mathbf{dx}_{2345} + \mathbf{dx}_{1357} - \mathbf{dx}_{1346} - \mathbf{dx}_{1256} - \mathbf{dx}_{1247}. \end{aligned}$$

Notice that Φ_0 is self-dual and that if we decompose $\mathbb{R}^8 = \mathbb{R} \oplus \mathbb{R}^7$ then

$$\Phi_0 = dx_0 \wedge \varphi_0 + *\varphi_0. \quad (1)$$

If we instead identify \mathbb{R}^8 with \mathbb{C}^4 such that the coordinates (z_1, z_2, z_3, z_4) on \mathbb{C}^4 are related to the real coordinates on \mathbb{R}^8 by $z_1 = x_0 + ix_1$, $z_2 = x_2 + ix_3$, $z_3 = x_4 + ix_5$, $z_4 = x_6 + ix_7$, then we can decompose Φ_0 as follows:

$$\Phi_0 = \frac{1}{2} \omega_0 \wedge \omega_0 + \text{Re } \Omega_0, \quad (2)$$

where ω_0 and Ω_0 are the standard Kähler form and holomorphic volume form on \mathbb{C}^4 respectively.

We can define the simple 14-dimensional exceptional Lie group G_2 as the stabilizer of φ_0 in $\text{GL}(7, \mathbb{R})$, leading us to refer to φ_0 as the ‘ G_2 3-form’ on \mathbb{R}^7 . Similarly, the stabilizer of Φ_0 in $\text{GL}(8, \mathbb{R})$ is the simple 21-dimensional Lie group $\text{Spin}(7)$, and we may call Φ_0 the ‘ $\text{Spin}(7)$ 4-form’ on \mathbb{R}^8 . Using φ_0 and Φ_0 we can now define general G_2 and $\text{Spin}(7)$ structures.

Definition 2.2 Let X be an oriented 7-manifold. We say that a 3-form φ on X is *positive* if, for all $x \in X$, $\varphi|_x = \iota_x^*(\varphi_0)$ for some orientation-preserving isomorphism $\iota_x : T_x X \rightarrow \mathbb{R}^7$. We denote the bundle of positive 3-forms $\Lambda_+^3 T^* X$.

A positive 3-form φ on X determines a unique metric g_φ on X such that, if $*\varphi$ is the Hodge dual to φ with respect to g_φ and $g_{\mathbb{R}^7}$ is the Euclidean metric on \mathbb{R}^7 , the triple $(\varphi, *\varphi, g_\varphi)$ is identified with $(\varphi_0, *\varphi_0, g_{\mathbb{R}^7})$ at each point. Hence, we may call a choice of $\varphi \in C^\infty(\Lambda_+^3 T^* X)$ a G_2 *structure* on X .

Definition 2.3 Let Y be an oriented 8-manifold. We say that a 4-form Φ on Y is *positive* if, for all $y \in Y$, $\Phi|_y = \iota_y^*(\Phi_0)$ for some orientation-preserving isomorphism $\iota_y : T_y Y \rightarrow \mathbb{R}^8$. We denote the bundle of positive 4-forms $\Lambda_+^4 T^* Y$.

A positive 4-form Φ determines a unique metric g_Φ on Y such that, if $g_{\mathbb{R}^8}$ is the Euclidean metric on \mathbb{R}^8 , the pair (Φ, g_Φ) is identified with $(\Phi_0, g_{\mathbb{R}^8})$ at each point. A choice of $\Phi \in C^\infty(\Lambda^4_+ T^*Y)$ is thus called a $\text{Spin}(7)$ *structure* on Y .

Since G_2 is the automorphism group of the octonions \mathbb{O} , there is a natural connection between the G_2 and $\text{Spin}(7)$ forms and certain structures on \mathbb{O} . The imaginary octonions are endowed with a cross product structure, which may be extended to a skew-symmetric product on \mathbb{O} , and one may further define triple and fourfold cross products on \mathbb{O} . We may identify $\text{Im } \mathbb{O}$ with \mathbb{R}^7 so that for all vectors $x, y, z \in \mathbb{R}^7$,

$$\varphi_0(x, y, z) = g_{\mathbb{R}^7}(x \times y, z).$$

Moreover, we may identify \mathbb{O} with \mathbb{R}^8 so that for vectors $x, y, z, w \in \mathbb{R}^8$,

$$\Phi_0(x, y, z, w) = g_{\mathbb{R}^8}(x \times y \times z, w).$$

These observations lead us to make the following definition.

Definition 2.4 Let X be an oriented 7-manifold with a G_2 structure φ . In the notation of Definition 2.2, we define a *cross product* on X via the equation

$$\varphi(x, y, z) = g_\varphi(x \times y, z)$$

for $x, y, z \in C^\infty(TX)$.

If Y is an oriented 8-manifold with a $\text{Spin}(7)$ structure Φ , we may similarly define a *triple cross product* on Y by:

$$\Phi(x, y, z, w) = g_\Phi(x \times y \times z, w),$$

where $x, y, z, w \in C^\infty(TY)$ and g_Φ is given in Definition 2.3.

Remarks

- (a) This relationship between G_2 structures and cross products allows us to deduce from [22, Corollary 3.3] that a 7-manifold admits a G_2 structure if and only if it is oriented and spin.
- (b) By [23, Theorem 3.4], the obstructions to the existence of a $\text{Spin}(7)$ structure on Y are not only given by the first two Stiefel–Whitney classes, but also by a certain combination of Euler and Pontryagin classes in $H^8(Y, \mathbb{Z})$.

We can distinguish certain G_2 structures as follows.

Definition 2.5 Let X be an oriented 7-manifold with a G_2 structure φ . We say that φ is *nearly parallel* if $d\varphi = 4\lambda * \varphi$ and $d*\varphi = 0$ for some constant $\lambda \neq 0$, and

we call (X, φ) a *nearly G_2 manifold*. If φ is closed and coclosed (i.e. if $\lambda = 0$), we say that the G_2 structure is *torsion-free* and (X, φ) is a G_2 manifold.

Remarks

- (a) By [39, Lemma 11.5], φ defines a torsion-free G_2 structure if and only if the holonomy of g_φ is contained in G_2 .
- (b) A nearly parallel G_2 structure φ can always be rescaled such that $\lambda = 1$. We shall therefore assume from now on that φ is chosen in this way.

We now define the G_2 structure on \mathcal{S}^7 .

Definition 2.6 Write $\mathbb{R}^8 \setminus \{0\} = \mathbb{R}^+ \times \mathcal{S}^7$ with r the coordinate on \mathbb{R}^+ . Then, since Φ_0 is self-dual, we may define a 3-form on \mathcal{S}^7 via the formula

$$\Phi_0|_{(r,p)} = r^3 dr \wedge \varphi|_p + r^4 * \varphi|_p, \quad (3)$$

where $*$ here denotes the usual Hodge star on \mathcal{S}^7 . Moreover, the fact that $d\Phi_0 = 0$ implies that $d\varphi = 4*\varphi$ and $d*\varphi = 0$. Since Φ_0 is a positive 4-form on \mathbb{R}^8 it follows that φ is a positive 3-form on \mathcal{S}^7 and g_φ given in Definition 2.2 is the round metric on \mathcal{S}^7 . Thus, (\mathcal{S}^7, φ) is a nearly G_2 manifold.

We may also define a contact structure on \mathcal{S}^7 using its relationship with \mathbb{C}^4 .

Definition 2.7 Consider $\mathbb{R}^+ \times \mathcal{S}^7 = \mathbb{R}^8 \setminus \{0\} \cong \mathbb{C}^4 \setminus \{0\}$, with r the coordinate on \mathbb{R}^+ , and let ω_0 be the usual Kähler form on \mathbb{C}^4 . Since ω_0 is a real 2-form and $d\omega_0 = 0$, there exists a 1-form γ on \mathcal{S}^7 such that

$$\omega_0|_{(r,p)} = r dr \wedge \gamma|_p + \frac{1}{2} r^2 d\gamma|_p. \quad (4)$$

Since $\omega_0^4 \neq 0$ we see that $\gamma \wedge (d\gamma)^3 \neq 0$, so γ defines a *contact structure* on \mathcal{S}^7 .

Remark One definition of a contact manifold is that the cone on it is symplectic: this is the content of Definition 2.7 in the special case of \mathcal{S}^7 .

As an aside we remark that, just as the $\text{Spin}(7)$ structure on \mathbb{R}^8 induces a G_2 structure on \mathcal{S}^7 , the G_2 structure on \mathbb{R}^7 induces an $\text{SU}(3)$ structure on \mathcal{S}^6 . In fact, \mathcal{S}^6 inherits a *nearly Kähler structure* from \mathbb{R}^7 . We do not discuss this structure fully as it is not necessary, but we define the *almost symplectic* and *almost complex* structures that \mathcal{S}^6 inherits from its embedding in \mathbb{R}^7 .

Definition 2.8 Write $\mathbb{R}^7 \setminus \{0\} = \mathbb{R}^+ \times \mathcal{S}^6$ with r the coordinate on \mathbb{R}^+ . Since $d\varphi_0 = 0$, there exists a 2-form ω on \mathcal{S}^6 such that

$$\varphi_0|_{(r,p)} = r^2 dr \wedge \omega|_p + \frac{1}{3} r^3 d\omega|_p.$$

Since φ_0 is positive, ω is non-degenerate but *not* closed. If $g_{\mathcal{S}^6}$ is the round metric on \mathcal{S}^6 , we can define an almost complex structure J on \mathcal{S}^6 via the formula $g_{\mathcal{S}^6}(Ju, v) = \omega(u, v)$ for tangent vectors u and v .

We say that an oriented surface Σ in \mathcal{S}^6 is a *pseudoholomorphic curve* if $\omega|_\Sigma = \text{vol}_\Sigma$ or, equivalently, if $J(T_\sigma \Sigma) = T_\sigma \Sigma$ for all $\sigma \in \Sigma$. We say that an oriented 3-dimensional submanifold L of \mathcal{S}^6 is *Lagrangian* if $\omega|_L \equiv 0$.

We shall now make some general observations about nearly G_2 manifolds for interest. We begin by defining $\text{Spin}(7)$ manifolds.

Definition 2.9 Let Y be an oriented 8-manifold with a $\text{Spin}(7)$ structure Φ . We say that the $\text{Spin}(7)$ structure is *torsion-free* if $d\Phi = 0$ and, in this case, we call (Y, Φ) a *$\text{Spin}(7)$ manifold*.

Remark By [39, Lemma 12.4], Φ defines a torsion-free $\text{Spin}(7)$ structure if and only if the holonomy of g_Φ is contained in $\text{Spin}(7)$.

As we have already seen, a simple way to see the relationship between various geometries is via *cones*, so we make the following convenient definition.

Definition 2.10 Given a Riemannian manifold (M, g) we define the *Riemannian cone* on M to be (CM, g_{CM}) where $CM = \mathbb{R}^+ \times M$ and $g_{CM} = dr^2 + r^2 g$, with r the coordinate on \mathbb{R}^+ .

We then make the following observation.

Lemma 2.11 *If X is a nearly G_2 manifold, CX is a $\text{Spin}(7)$ manifold. Conversely, if (X, g_X) is a Riemannian 7-manifold and CX is a $\text{Spin}(7)$ manifold then there exists a nearly parallel G_2 structure φ on X such that $g_X = g_\varphi$.*

Proof: By [1, Theorem 5], CX is a $\text{Spin}(7)$ manifold if and only if X possesses a Killing spinor ψ such that $\nabla_v \psi = -\frac{1}{2} v \cdot \psi$ for all tangent vector fields v . Further, by [20, Proposition 3.12], such a Killing spinor exists on X if and only if there is a nearly parallel G_2 structure on X whose associated metric is g_X . \square

Nearly parallel G_2 structures are discussed in some detail in [20] and many examples are given. In particular, they include some familiar structures which we now recall.

Definition 2.12 A Riemannian manifold M is *Sasakian* if and only if CM is Kähler. A Sasakian manifold is *Sasaki–Einstein* if CM is Ricci-flat, and it is *3-Sasakian* if CM is hyperkähler.

It is well-known that $\text{Spin}(7)$ metrics are Ricci-flat. Furthermore, Ricci-flat Kähler and hyperkähler 8-manifolds are examples of $\text{Spin}(7)$ manifolds. Thus, we have the following.

Lemma 2.13 *Sasaki–Einstein and 3-Sasakian 7-manifolds are both examples of nearly G_2 manifolds.*

Motivated by connections with work in high energy theoretical physics, infinite families of explicit Sasaki–Einstein manifolds M^{2n+1} are constructed in [21] for all $n \geq 2$. Thus, there is a wealth of concrete examples of nearly G_2 manifolds.

We conclude this section with a result concerning the generality of nearly parallel G_2 structures.

Lemma 2.14 *Nearly parallel G_2 structures enjoy the same local generality as torsion-free G_2 structures; namely, they locally depend on 6 functions of 6 variables up to diffeomorphism.*

Proof: In [8, Proposition 2], Bryant studied an exterior differential system defining the torsion-free G_2 structures and showed that it was involutive with certain characters. From this, Bryant was able to describe the local generality of torsion-free G_2 structures, up to diffeomorphism. Using similar calculations we are able to show that the corresponding exterior differential system defining nearly parallel G_2 structures is involutive with the same characters. \square

Remarks One should take results like Lemma 2.14 with a pinch of salt: it only proves the *local* existence of many nearly parallel G_2 structures. The analogue of Lemma 2.14 in 6 dimensions, namely that nearly Kähler structures on 6-manifolds have the same local generality as Calabi–Yau structures on complex 3-manifolds, currently appears to be misleading, given the large number of Calabi–Yau 3-folds versus the handful of known examples of nearly Kähler 6-manifolds.

3 Associative submanifolds of \mathcal{S}^7

In this section we define and discuss our primary objects of interest.

Definition 3.1 Let X be an oriented 7-manifold with a G_2 structure φ . Using the notation of Definition 2.2, define a vector-valued 3-form χ on X via the formula

$$*\varphi(x, y, z, w) = g_\varphi(\chi(x, y, z), w)$$

for vector fields x, y, z, w on X .

- (a) An oriented 3-fold A in X is *associative* if $\varphi|_A = \text{vol}_A$. Equivalently, A is associative if $\chi|_A \equiv 0$ and $\varphi|_A > 0$.
- (b) An oriented 4-fold N in X is *coassociative* if $*\varphi|_N = \text{vol}_N$. Equivalently, N is coassociative if $\varphi|_N \equiv 0$ and $*\varphi|_N > 0$.

Remarks Since the orthogonal complement of an associative 3-plane in \mathbb{R}^7 is a coassociative 4-plane, one can equivalently define an oriented 3-fold A in (X, φ) to be associative if φ vanishes on the normal bundle of A . This characterisation shows that the apparently overdetermined system of equations given by the vanishing of χ is in fact determined. Though this property makes this alternative definition more attractive, it is not standard and we elect not to adopt it here.

We make an elementary observation.

Lemma 3.2 *There are no coassociative submanifolds of a nearly G_2 manifold.*

Proof: Suppose N is coassociative in a nearly G_2 manifold (X, φ) . Then $\varphi|_N \equiv 0$ implies that $d\varphi|_N \equiv 0$. However, $d\varphi = 4*\varphi$, so $*\varphi|_N \equiv 0$, yielding our required contradiction. \square

We may define distinguished submanifolds of a manifold with a $\text{Spin}(7)$ structure as follows.

Definition 3.3 Let Y be an oriented 8-manifold with a $\text{Spin}(7)$ structure Φ . An oriented 4-fold N in Y is *Cayley* if $\Phi|_N = \text{vol}_N$.

We can alternatively characterise Cayley 4-folds as the 4-dimensional submanifolds of Y on which a certain vector-valued 4-form τ vanishes. This 4-form arises from the imaginary part of the fourfold cross product on the octonions. We describe τ in the special case when $Y = \mathbb{R}^8$.

Definition 3.4 Use the notation of Definition 2.1. Define a vector-valued 4-form τ on \mathbb{R}^8 such that, if τ_j is the component of τ in the direction $\frac{\partial}{\partial x_j}$, then:

$$\begin{aligned}\tau_0 &= \Phi_0; \\ \tau_1 &= dx_{0247} + dx_{0256} + dx_{0346} - dx_{0357} + dx_{1246} - dx_{1257} - dx_{1347} - dx_{1356}; \\ \tau_2 &= -dx_{0147} - dx_{0156} - dx_{0345} - dx_{0367} - dx_{1245} - dx_{1267} - dx_{2347} - dx_{2356}; \\ \tau_3 &= -dx_{0146} + dx_{0157} + dx_{0245} + dx_{0267} - dx_{1345} - dx_{1367} - dx_{2346} + dx_{2357}; \\ \tau_4 &= dx_{0127} + dx_{0136} - dx_{0235} - dx_{0567} - dx_{1234} - dx_{1467} + dx_{2457} + dx_{3456}; \\ \tau_5 &= dx_{0126} - dx_{0137} + dx_{0234} + dx_{0467} - dx_{1235} - dx_{1567} + dx_{2456} - dx_{3457}; \\ \tau_6 &= -dx_{0125} - dx_{0134} - dx_{0237} - dx_{0457} - dx_{1236} - dx_{1456} - dx_{2567} - dx_{3467}; \\ \tau_7 &= -dx_{0124} + dx_{0135} + dx_{0236} + dx_{0456} - dx_{1237} - dx_{1457} - dx_{2467} + dx_{3567}.\end{aligned}$$

The forms τ_j correspond to the components of the fourfold cross product on \mathbb{O} discussed in [25, Appendix IV.B]. Thus, by [25, Corollary IV.1.29], we have that an oriented 4-fold N in \mathbb{R}^8 is Cayley if and only if $\tau_j|_N \equiv 0$ for $j = 1, \dots, 7$ (up to a choice of orientation so that $\Phi_0|_N = \tau_0|_N > 0$).

If we identify \mathbb{R}^8 with \mathbb{C}^4 as in Definition 2.1 we see that:

$$\begin{aligned}\tau_1 &= \text{Im } \Omega_0; \\ \tau_2 + i\tau_3 &= i(dz_1 \wedge dz_2 - d\bar{z}_3 \wedge d\bar{z}_4) \wedge \omega_0; \\ \tau_4 + i\tau_5 &= i(dz_1 \wedge dz_3 - d\bar{z}_4 \wedge d\bar{z}_2) \wedge \omega_0; \\ \tau_6 + i\tau_7 &= i(dz_1 \wedge dz_4 - d\bar{z}_2 \wedge d\bar{z}_3) \wedge \omega_0.\end{aligned}$$

In particular, we can confirm that $\omega_0|_N \equiv 0$ and $\text{Im } \Omega_0|_N \equiv 0$ force $\tau_j|_N \equiv 0$ for $j = 1, \dots, 7$; i.e. if N is *special Lagrangian* in \mathbb{C}^4 then it is Cayley in \mathbb{R}^8 .

We now give our first basic result concerning associative submanifolds of \mathcal{S}^7 .

Lemma 3.5 *Let X be a nearly G_2 manifold and recall Definition 2.10. A 4-dimensional cone $N = \mathbb{R}^+ \times A$ in CX is Cayley if and only if A is associative in X . In particular, the link of a Cayley cone in \mathbb{R}^8 is associative in \mathcal{S}^7 .*

Proof: We show this when $X = \mathcal{S}^7$, but the general case is similar. By Definition 3.3, N is a Cayley cone in \mathbb{R}^8 if and only if $\Phi_0|_N = \text{vol}_N$, where Φ_0 is given in Definition 2.1. By (3), $\Phi_0|_N = \text{vol}_N$ if and only if $\varphi|_A = \text{vol}_A$, where φ is the G_2 structure on \mathcal{S}^7 . The result follows from Definition 3.1. \square

From Lemma 3.5 we may deduce the following.

Corollary 3.6 *Associative submanifolds of \mathcal{S}^7 are minimal and real analytic wherever they are non-singular.*

Proof: This is immediate from the fact that Cayley 4-folds in \mathbb{R}^8 have these properties by [25, Theorem II.4.2] and [29, Theorem 12.4.3]. \square

Given the real analyticity of associative 3-folds in \mathcal{S}^7 , we can apply the Cartan–Kähler Theorem as in [25, Theorem IV.4.1] to prove the following.

Proposition 3.7 *Let S be an oriented real analytic surface in \mathcal{S}^7 . There locally exists a locally unique associative 3-fold in \mathcal{S}^7 containing S .*

We deduce that associative 3-folds in \mathcal{S}^7 locally depend on 4 functions of 2 variables, in the sense of exterior differential systems.

Using the relationship between Cayley geometry and other geometries we can give some examples of associative 3-folds in \mathcal{S}^7 . We begin with the following.

Proposition 3.8 *Decompose $\mathbb{R}^8 = \mathbb{R} \oplus \mathbb{R}^7$ and recall Definition 2.8.*

(a) *Let Σ be an oriented surface in \mathcal{S}^6 and let*

$$A = \{(\cos t, \sigma \sin t) \in \mathbb{R} \oplus \mathbb{R}^7 : \sigma \in \Sigma, t \in (0, \pi)\} \subseteq \mathcal{S}^7.$$

Then A is an associative 3-fold in \mathcal{S}^7 if and only if Σ is a pseudoholomorphic curve in \mathcal{S}^6 .

(b) *Let L be an oriented 3-dimensional submanifold of \mathcal{S}^6 and let $A = \{0\} \times L \subseteq \mathcal{S}^7$. Then A is an associative 3-fold in \mathcal{S}^7 if and only if L is a Lagrangian submanifold of \mathcal{S}^6 .*

Proof: Let C be a 4-dimensional submanifold in $\mathbb{R}^8 = \mathbb{R} \oplus \mathbb{R}^7$. Then $C = \mathbb{R} \times N$ is Cayley if and only if N is associative in \mathbb{R}^7 by (1) and Definitions 3.1(a) and 3.3. Similarly, $C = \{0\} \times N$ is Cayley if and only if N is coassociative in \mathbb{R}^7 by (1) and Definitions 3.1(b) and 3.3. Finally, we note that a cone N in \mathbb{R}^7 is either associative or coassociative if and only if the link of N in \mathcal{S}^6 is either a pseudoholomorphic curve or a Lagrangian submanifold respectively by Definitions 2.8 and 3.1. \square

Many examples of pseudoholomorphic curves and Lagrangians in \mathcal{S}^6 are known (see, for example, [4] and [34]), thus providing examples of associative 3-folds in \mathcal{S}^7 . Pseudoholomorphic curves in \mathcal{S}^6 depend locally on 4 functions of 1 variable, whilst Lagrangians in \mathcal{S}^6 depend on 2 functions of 2 variables. This suggests that the generic associative 3-fold in \mathcal{S}^7 is not constructed purely from these submanifolds of \mathcal{S}^6 .

Proposition 3.9 *Recall the contact structure γ on \mathcal{S}^7 given in Definition 2.7 and that $A^3 \subseteq \mathcal{S}^7$ is called Legendrian if $\gamma|_A \equiv 0$. A minimal Legendrian submanifold of \mathcal{S}^7 is associative.*

Proof: Let A be Legendrian in \mathcal{S}^7 . By (4), the cone CA is Lagrangian in $\mathbb{C}^4 \cong \mathbb{R}^8$. If A is minimal, then CA is minimal and by [26, Proposition 2.5] is, in fact, *special Lagrangian with phase $e^{i\theta}$* in \mathbb{C}^4 for some constant θ ; that is, in the notation of Definition 2.1, $\omega_0|_{CA} \equiv 0$ and $\text{Re}(e^{-i\theta}\Omega_0)|_{CA} = \text{vol}_{CA}$. By making a suitable identification of $\mathbb{C}^4 \cong \mathbb{R}^8$ we can ensure that $e^{i\theta} = 1$, and so CA is Cayley in \mathbb{R}^8 by (2) and Definition 3.3. The result follows from Lemma 3.5. \square

From the proof of Proposition 3.9, we have that minimal Legendrian submanifolds of \mathcal{S}^7 are the links of special Lagrangian cones in \mathbb{C}^4 . Explicit examples of such cones in \mathbb{C}^4 are given in [29, Examples 8.3.5 & 8.3.6]. Moreover, a gluing construction is described in [27], [28] which, in particular, yields infinitely many topological types of minimal Legendrian, hence associative, submanifolds of \mathcal{S}^7 .

We may also connect associative 3-folds in \mathcal{S}^7 to complex geometry in the following manner.

Proposition 3.10 *Let $\mathbf{u} : \Sigma \rightarrow \mathbb{CP}^3$ be a holomorphic curve. The Hopf fibration of \mathcal{S}^7 over \mathbb{CP}^3 induces a circle bundle $\mathcal{C}(\Sigma)$ over Σ . Let $\mathbf{x} : \mathcal{C}(\Sigma) \rightarrow \mathcal{S}^7$ be such that the following diagram commutes:*

$$\begin{array}{ccc} \mathcal{C}(\Sigma) & \xrightarrow{\mathbf{x}} & \mathcal{S}^7 \\ \downarrow & & \downarrow \\ \Sigma & \xrightarrow{\mathbf{u}} & \mathbb{CP}^3. \end{array}$$

Then $\mathbf{x}(\mathcal{C}(\Sigma))$ is an associative 3-fold in \mathcal{S}^7 .

Proof: A holomorphic curve in \mathbb{CP}^3 is the (complex) link of a complex cone in \mathbb{C}^4 . Complex surfaces S in \mathbb{C}^4 are Cayley in \mathbb{R}^8 by (2) and Definition 3.3 since, in the notation of Definition 2.1, $\frac{1}{2}\omega_0 \wedge \omega_0|_S = \text{vol}_S$ and $\Omega_0|_S \equiv 0$. The result follows from Lemma 3.5. \square

Remark Proposition 3.10 simply states that the (real) link in \mathcal{S}^7 of a complex 2-dimensional cone in \mathbb{C}^4 is associative.

Holomorphic curves in \mathbb{CP}^3 depend locally on 4 functions of 1 variable and minimal Legendrian submanifolds of \mathcal{S}^7 depend locally on 2 functions of 2 variables, so we would again expect that the generic associative 3-fold in \mathcal{S}^7 does not arise from these geometries.

Remarks There is some overlap between the examples of associative 3-folds in \mathcal{S}^7 given by Propositions 3.8-3.10. In particular, we have the following.

- (a) A minimal Legendrian surface in a totally geodesic \mathcal{S}^5 in \mathcal{S}^6 is a pseudoholomorphic curve, and so defines an associative 3-fold by Proposition 3.8(a) which is also minimal Legendrian in \mathcal{S}^7 .
- (b) A holomorphic curve in a totally geodesic \mathbb{CP}^2 in \mathbb{CP}^3 defines an associative 3-fold by Proposition 3.10 which lies in a totally geodesic \mathcal{S}^6 and so is Lagrangian by Proposition 3.8(b).

One might ask about the relationship between associative geometry and the Hopf fibration $\mathcal{S}^3 \hookrightarrow \mathcal{S}^7 \rightarrow \mathcal{S}^4$, which results from viewing \mathcal{S}^7 as the unit sphere in \mathbb{H}^2 , where \mathbb{H} is the quaternions, and $\mathcal{S}^4 \cong \mathbb{HP}^1$. The projection $\pi(A)$ of an associative 3-fold A in \mathcal{S}^7 to \mathcal{S}^4 is either a point or a surface. If $\pi(A)$ is a point, A is a totally geodesic \mathcal{S}^3 . If $\pi(A)$ is a surface, then it is equal to the projection of A under the fibration $\mathcal{S}^1 \hookrightarrow \mathcal{S}^7 \rightarrow \mathbb{CP}^3 \rightarrow \mathcal{S}^4$. Thus A must be the Hopf lift of a *horizontal* holomorphic curve in \mathbb{CP}^3 . We can view this horizontal holomorphic curve as a “twistor lift” of the surface $\pi(A)$ in \mathcal{S}^4 to \mathbb{CP}^3 (c.f. [7]).

4 The structure equations

To study associative submanifolds of \mathcal{S}^7 we shall think of the 7-sphere as the homogeneous space $\text{Spin}(7)/\text{G}_2$. Since we are considering \mathcal{S}^7 with its G_2 structure, we may view $\text{Spin}(7)$ as the G_2 frame bundle over \mathcal{S}^7 . Therefore, we shall need the structure equations of $\text{Spin}(7)$.

We begin by recalling the following result [6, Proposition 1.1].

Proposition 4.1 *Extend the elements of $\text{Spin}(7) \subseteq \text{End}(\mathbb{O})$ complex linearly so that $\text{Spin}(7) \subseteq \text{End}(\mathbb{C} \otimes_{\mathbb{R}} \mathbb{O})$. Using the standard basis of $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{O}$ to represent $\text{End}_{\mathbb{C}}(\mathbb{C} \otimes_{\mathbb{R}} \mathbb{O})$ as the 8×8 complex-valued matrices, the Lie algebra $\mathfrak{spin}(7)$ of $\text{Spin}(7)$ has the following matrix presentation:*

$$\mathfrak{spin}(7) = \left\{ \begin{pmatrix} i\rho & -\bar{\mathfrak{h}}^T & 0 & -\theta^T \\ \mathfrak{h} & \kappa & \theta & [\bar{\theta}] \\ 0 & -\bar{\theta}^T & -i\rho & -\mathfrak{h}^T \\ \bar{\theta} & [\theta] & \bar{\mathfrak{h}} & \bar{\kappa} \end{pmatrix} : \begin{array}{l} \mathfrak{h}, \theta \in M_{3 \times 1}(\mathbb{C}), \\ \rho \in \mathbb{R}, \kappa \in M_{3 \times 3}(\mathbb{C}), \\ \kappa = -\bar{\kappa}^T, \text{Tr } \kappa = -i\rho \end{array} \right\}$$

where

$$[(x \ y \ z)^T] = \begin{pmatrix} 0 & z & -y \\ -z & 0 & x \\ y & -x & 0 \end{pmatrix}. \quad (5)$$

This presentation of $\mathfrak{spin}(7)$ is not currently conducive to the study of associative 3-folds. We thus make the following substitutions:

$$\mathfrak{h} = \frac{1}{2} \begin{pmatrix} \omega_1 + \alpha_1 + i(\omega_2 + \alpha_2) \\ \beta_3^5 + \frac{4}{3}\eta_6 + i(-\beta_3^6 + \frac{4}{3}\eta_5) \\ \beta_3^4 + \frac{4}{3}\eta_7 + i(-\beta_3^7 + \frac{4}{3}\eta_4) \end{pmatrix}; \quad (6)$$

$$\theta = \frac{1}{2} \begin{pmatrix} \omega_1 - \alpha_1 + i(\omega_2 - \alpha_2) \\ -\beta_3^5 + \frac{2}{3}\eta_6 + i(\beta_3^6 + \frac{2}{3}\eta_5) \\ -\beta_3^4 + \frac{2}{3}\eta_7 + i(\beta_3^7 + \frac{2}{3}\eta_4) \end{pmatrix}; \quad (7)$$

$$\kappa_{11} = -i\alpha_3; \quad \kappa_{22} = \frac{i}{2}(\alpha_3 - \omega_3 + \gamma_3); \quad \kappa_{33} = \frac{i}{2}(\alpha_3 - \omega_3 - \gamma_3); \quad (8)$$

$$\kappa_{21} = \frac{1}{2}(\beta_1^6 + \beta_2^5) + \frac{i}{2}(\beta_1^5 - \beta_2^6); \quad \kappa_{31} = \frac{1}{2}(\beta_1^7 + \beta_2^4) + \frac{i}{2}(\beta_1^4 - \beta_2^7); \quad (9)$$

$$\kappa_{32} = -\frac{1}{2}\gamma_1 - \frac{i}{2}\gamma_2; \quad \rho = \omega_3, \quad (10)$$

for real numbers $\omega_j, \alpha_j, \gamma_j, \eta_a, \beta_j^a$ for $j = 1, 2, 3$ and $a = 4, 5, 6, 7$ such that

$$\beta_1^4 + \beta_2^7 + \beta_3^6 = 0, \quad \beta_1^5 + \beta_2^6 - \beta_3^7 = 0, \quad (11)$$

$$\beta_1^6 - \beta_2^5 - \beta_3^4 = 0, \quad \beta_1^7 - \beta_2^4 + \beta_3^5 = 0. \quad (12)$$

We deduce the following.

Proposition 4.2 *Let the index j range from 1 to 3 and the index a range from 4 to 7. We may write the Lie algebra $\mathfrak{spin}(7)$ of $\text{Spin}(7) \subseteq \text{GL}(8, \mathbb{R})$ as:*

$$\mathfrak{spin}(7) = \left\{ \begin{pmatrix} 0 & -\omega^T & -\eta^T \\ \omega & [\alpha] & -\beta^T - \frac{1}{3}\{\eta\}^T \\ \eta & \beta + \frac{1}{3}\{\eta\} & \frac{1}{2}[\alpha - \omega]_+ + \frac{1}{2}[\gamma]_- \end{pmatrix} : \begin{array}{l} \omega = (\omega_j), \alpha = (\alpha_j), \\ \gamma = (\gamma_j) \in \text{M}_{3 \times 1}(\mathbb{R}), \\ \eta = (\eta_a) \in \text{M}_{4 \times 1}(\mathbb{R}), \\ \beta = (\beta_j^a) \in \text{M}_{4 \times 3}(\mathbb{R}), \end{array} \right. \\ \left. \begin{array}{l} \beta_1^4 + \beta_2^7 + \beta_3^6 = 0, \quad \beta_1^5 + \beta_2^6 - \beta_3^7 = 0, \\ \beta_1^6 - \beta_2^5 - \beta_3^4 = 0, \quad \beta_1^7 - \beta_2^4 + \beta_3^5 = 0 \end{array} \right\},$$

where $[(x \ y \ z)^T]$ is defined in (5),

$$[(x \ y \ z)^T]_+ = \begin{pmatrix} 0 & -x & -y & z \\ x & 0 & z & y \\ y & -z & 0 & -x \\ -z & -y & x & 0 \end{pmatrix}, \quad (13)$$

$$[(x \ y \ z)^T]_- = \begin{pmatrix} 0 & -x & -y & -z \\ x & 0 & z & -y \\ y & -z & 0 & x \\ z & y & -x & 0 \end{pmatrix} \quad (14)$$

and

$$\{(p \ q \ r \ s)^T\} = \begin{pmatrix} -q & -r & s \\ p & s & r \\ -s & p & -q \\ r & -q & -p \end{pmatrix} \quad (15)$$

The notation $[\]_{\pm}$ reflects the splitting of $\mathfrak{so}(4) \cong \Lambda^2(\mathbb{R}^4)^*$ into positive and negative subspaces. The conditions on β and the symmetries of $\{ \}$ in (15) are related to the cross product on $\text{Im } \mathbb{O}$ defined by φ_0 given in Definition 2.1.

Let $g : \text{Spin}(7) \rightarrow \text{GL}(8, \mathbb{R})$ take $\text{Spin}(7)$ to the identity component of the Lie subgroup of $\text{GL}(8, \mathbb{R})$ with Lie algebra $\mathfrak{spin}(7)$. Write $g = (\mathbf{x} \ \mathbf{e} \ \mathbf{f})$ where, for each $p \in \text{Spin}(7)$, $\mathbf{x}(p) \in M_{8 \times 1}(\mathbb{R})$, $\mathbf{e}(p) = (\mathbf{e}_1 \ \mathbf{e}_2 \ \mathbf{e}_3)(p) \in M_{8 \times 3}(\mathbb{R})$ and $\mathbf{f}(p) = (\mathbf{f}_4 \ \mathbf{f}_5 \ \mathbf{f}_6 \ \mathbf{f}_7)(p) \in M_{8 \times 4}(\mathbb{R})$. The Maurer–Cartan form $\phi = g^{-1}dg$ takes values in $\mathfrak{spin}(7)$, so it can be written as

$$\phi = \begin{pmatrix} 0 & -\omega^T & -\eta^T \\ \omega & [\alpha] & -\beta^T - \frac{1}{3}\{\eta\}^T \\ \eta & \beta + \frac{1}{3}\{\eta\} & \frac{1}{2}[\alpha - \omega]_+ + \frac{1}{2}[\gamma]_- \end{pmatrix} \quad (16)$$

for appropriate matrix-valued 1-forms ω , η , α , β and γ . Moreover, if \times is the cross product on \mathcal{S}^7 determined by the G_2 structure as in Definition 2.4, then the symmetries of β given in (11)-(12) can be expressed neatly as

$$\sum_{i=1}^3 \mathbf{e}_i \times (\mathbf{f}\beta)_j = 0 \quad \text{for } j = 1, 2, 3,$$

using the obvious notation for components of $\mathbf{f}\beta$.

An associative 3-fold A in \mathcal{S}^7 may be lifted to $\text{Spin}(7)$ by choosing suitable adapted G_2 frames on A . It is clear that we may adapt frames such that \mathbf{x} , \mathbf{e} and ω are identified with a point in A , an orthonormal frame and orthonormal coframe for A respectively. We thus recognise \mathbf{f} as an orthonormal frame for the normal space to A in \mathcal{S}^7 and see that η vanishes on A .

From $dg = g\phi$ and the Maurer–Cartan equation $d\phi + \phi \wedge \phi = 0$, we derive the *first and second structure equations*.

Proposition 4.3 *Let $g = (\mathbf{x} \ \mathbf{e} \ \mathbf{f}) : \text{Spin}(7) \rightarrow \text{GL}(8, \mathbb{R})$ as described above. Write $\phi = g^{-1}dg$ as in (16) such that ϕ takes values in $\mathfrak{spin}(7)$. Recall $[\]_{\pm}$ and $\{ \}$ given in (13)-(15). The first structure equations of $\text{Spin}(7)$ are:*

$$\begin{aligned} d\mathbf{x} &= \mathbf{e}\omega + \mathbf{f}\eta; \\ d\mathbf{e} &= -\mathbf{x}\omega^T + \mathbf{e}[\alpha] + \mathbf{f}(\beta + \frac{1}{3}\{\eta\}); \\ d\mathbf{f} &= -\mathbf{x}\eta^T - \mathbf{e}(\beta + \frac{1}{3}\{\eta\})^T + \frac{1}{2}\mathbf{f}([\alpha - \omega]_+ + [\gamma]_-). \end{aligned}$$

On the adapted frame bundle of an associative 3-fold A in \mathcal{S}^7 , $\mathbf{x} : A \rightarrow \mathcal{S}^7$ and $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{f}_4, \mathbf{f}_5, \mathbf{f}_6, \mathbf{f}_7\}$ is a local oriented orthonormal basis for $TA \oplus NA$. Thus, the first structure equations become:

$$d\mathbf{x} = \mathbf{e}\omega; \quad (17)$$

$$d\mathbf{e} = -\mathbf{x}\omega^T + \mathbf{e}[\alpha] + \mathbf{f}\beta; \quad (18)$$

$$d\mathbf{f} = -\mathbf{e}\beta^T + \frac{1}{2}\mathbf{f}([\alpha - \omega]_+ + [\gamma]_-). \quad (19)$$

Proposition 4.4 Use the notation of Proposition 4.3 and recall $[\]_{\pm}$ and $\{ \}$ given in (13)-(15). The second structure equations of $\text{Spin}(7)$ are:

$$\begin{aligned} d\omega &= -[\alpha] \wedge \omega + (\beta + \frac{1}{3}\{\eta\})^T \wedge \eta; \\ d\eta &= -(\beta + \frac{1}{3}\{\eta\}) \wedge \omega - \frac{1}{2}([\alpha - \omega]_+ + [\gamma]_-) \wedge \eta; \\ d[\alpha] &= -[\alpha] \wedge [\alpha] + \omega \wedge \omega^T + (\beta + \frac{1}{3}\{\eta\})^T \wedge (\beta + \frac{1}{3}\{\eta\}); \\ d(\beta + \frac{1}{3}\{\eta\}) &= \eta \wedge \omega^T - (\beta + \frac{1}{3}\{\eta\}) \wedge [\alpha] \\ &\quad - \frac{1}{2}([\alpha - \omega]_+ + [\gamma]_-) \wedge (\beta + \frac{1}{3}\{\eta\}); \\ \frac{1}{2}d([\alpha - \omega]_+ + [\gamma]_-) &= -\frac{1}{4}[\alpha - \omega]_+ \wedge [\alpha - \omega]_+ - \frac{1}{4}[\gamma]_- \wedge [\gamma]_- \\ &\quad + \eta \wedge \eta^T + (\beta + \frac{1}{3}\{\eta\}) \wedge (\beta + \frac{1}{3}\{\eta\})^T. \end{aligned}$$

On the adapted frame bundle of an associative 3-fold in \mathcal{S}^7 , there exists a local tensor of functions $h = h_{jk}^a = h_{kj}^a$, for $1 \leq j, k \leq 3$ and $4 \leq a \leq 7$, such that:

$$d\omega = -[\alpha] \wedge \omega; \quad (20)$$

$$\beta = h\omega; \quad (21)$$

$$d[\alpha] = -[\alpha] \wedge [\alpha] + \omega \wedge \omega^T + \beta^T \wedge \beta; \quad (22)$$

$$d\beta = -\beta \wedge [\alpha] - \frac{1}{2}([\alpha - \omega]_+ + [\gamma]_-) \wedge \beta; \quad (23)$$

$$\frac{1}{2}d([\alpha - \omega]_+ + [\gamma]_-) = -\frac{1}{4}[\alpha - \omega]_+ \wedge [\alpha - \omega]_+ - \frac{1}{4}[\gamma]_- \wedge [\gamma]_- + \beta \wedge \beta^T. \quad (24)$$

We see from (18) and (20) that $[\alpha]$ defines the Levi-Civita connection of an associative 3-fold. Moreover, (19) shows that $\frac{1}{2}([\alpha - \omega]_+ + [\gamma]_-)$ defines the induced connection on the normal bundle of A in \mathcal{S}^7 .

Notes If we set $\mathbf{h} = (\mathbf{v}_0 \ \mathbf{v}_1 \ \mathbf{v}_2 \ \mathbf{v}_3 \ \bar{\mathbf{v}}_0 \ \bar{\mathbf{v}}_1 \ \bar{\mathbf{v}}_2 \ \bar{\mathbf{v}}_3)$ where

$$\mathbf{v}_0 = \frac{1}{2}(\mathbf{x} - i\mathbf{e}_3), \quad \mathbf{v}_1 = \frac{1}{2}(\mathbf{e}_1 - i\mathbf{e}_2), \quad \mathbf{v}_2 = \frac{1}{2}(\mathbf{f}_6 - i\mathbf{f}_5), \quad \mathbf{v}_3 = \frac{1}{2}(\mathbf{f}_7 - i\mathbf{f}_4),$$

then $\mathbf{h}^{-1}d\mathbf{h} = \psi$ takes values in $\mathfrak{spin}(7)$ as given in Proposition 4.1 and may be written as

$$\psi = \begin{pmatrix} i\rho & -\bar{\mathbf{h}}^T & 0 & -\theta^T \\ \mathbf{h} & \kappa & \theta & [\bar{\theta}] \\ 0 & -\bar{\theta}^T & -i\rho & -\mathbf{h}^T \\ \bar{\theta} & [\theta] & \bar{\mathbf{h}} & \bar{\kappa} \end{pmatrix},$$

where \mathfrak{h} , θ , κ and ρ are given by inverting our substitutions (6)-(10). Thus, we can recover the structure equations for $\text{Spin}(7)$ given in [6] from $d\mathfrak{h} = \mathfrak{h}\psi$ and $d\psi + \psi \wedge \psi = 0$. This form for the structure equations will be invaluable in §7-9.

4.1 The second fundamental form

The equations (18) and (21) show that β encodes the *second fundamental form* of A , essentially given by the tensor of functions h . We discuss this formally.

Definition 4.5 Let A be an associative 3-fold in \mathcal{S}^7 and use the notation of Propositions 4.3-4.4. The *second fundamental form* $\Pi_A \in C^\infty(S^2T^*A; NA)$ of A can be written locally, using summation notation, as

$$\Pi_A = h_{jk}^a \mathbf{f}_a \otimes \omega_j \omega_k$$

for a tensor of functions h such that $h_{jk}^a = h_{kj}^a$, $a = 4, 5, 6, 7$, $j, k = 1, 2, 3$. The tensor h is the same tensor that appears in (21) and satisfies the further symmetry conditions for all j :

$$h_{1j}^4 + h_{2j}^7 + h_{3j}^6 = 0; \quad h_{1j}^5 + h_{2j}^6 - h_{3j}^7 = 0; \quad (25)$$

$$h_{1j}^6 - h_{2j}^5 - h_{3j}^4 = 0; \quad h_{1j}^7 - h_{2j}^4 + h_{3j}^5 = 0. \quad (26)$$

These symmetry conditions follow from (11)-(12) and are equivalent to:

$$\mathbf{e}_i \times \Pi_A(\mathbf{e}_i, \mathbf{e}_j) = 0$$

for $j = 1, 2, 3$, using summation notation and the cross product on \mathcal{S}^7 .

Remark The symmetry conditions (25)-(26) imply that associative 3-folds in \mathcal{S}^7 are minimal. This confirms our earlier result in Corollary 3.6.

We can interpret (22) as the *Gauss equation* for an associative 3-fold A in \mathcal{S}^7 and (23) as the *Codazzi equation*; i.e. (22) relates the Riemann curvature to the second fundamental form Π_A and (23) imposes conditions on the derivative of Π_A . We can also view (24) as the *Ricci equation* relating the curvature of the normal connection to Π_A . In particular, if δ_{ij} is the Kronecker delta tensor and we use summation notation, the Riemann curvature $\frac{1}{2}R_{ijkl}\omega_k \wedge \omega_l$ and the curvature of the normal connection $\frac{1}{2}R_{abkl}^\perp \omega_k \wedge \omega_l$ satisfy:

$$R_{ijkl} = h_{ik}^a h_{jl}^a - h_{il}^a h_{jk}^a + \delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk} \quad \text{and} \quad R_{abkl}^\perp = h_{ik}^a h_{il}^b - h_{il}^a h_{ik}^b.$$

To simplify some of our calculations later, we have the following useful result.

Proposition 4.6 *Let A be an associative 3-fold in \mathcal{S}^7 . There exist a local orthonormal coframing $\{\omega_1, \omega_2, \omega_3\}$ for A and framing $\{\mathbf{f}_4, \mathbf{f}_5, \mathbf{f}_6, \mathbf{f}_7\}$ for NA such that the second fundamental form Π_A of A is given by:*

$$\begin{aligned}\Pi_A = & \mathbf{f}_4 \otimes (2r\omega_3^2 - (r-s)\omega_1^2 - (r+s)\omega_2^2 + 2(t+w)\omega_1\omega_2) \\ & + \mathbf{f}_5 \otimes (v\omega_1^2 - v\omega_2^2 + 2u\omega_1\omega_2 + 4w\omega_3\omega_1 - 2r\omega_2\omega_3) \\ & + \mathbf{f}_6 \otimes (u\omega_1^2 - u\omega_2^2 - 2v\omega_1\omega_2 + 2r\omega_3\omega_1 - 4w\omega_2\omega_3) \\ & + \mathbf{f}_7 \otimes ((t-w)\omega_1^2 - (t-w)\omega_2^2 - 2s\omega_1\omega_2)\end{aligned}\quad (27)$$

for some local functions r, s, t, u, v, w on A , with $r \geq 0$. Moreover, if $w \equiv 0$ we can adapt frames such that $v \equiv 0$.

Proof: We define the choice of local orthonormal framing $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{f}_4, \mathbf{f}_5, \mathbf{f}_6, \mathbf{f}_7\}$ for $T\mathcal{S}^7|_A = TA \oplus NA$. Recall the cross product on \mathcal{S}^7 given by Definition 2.4.

Let g be the round metric on \mathcal{S}^7 and consider the function

$$G_1(\mathbf{e}, \mathbf{f}) = g(\Pi_A(\mathbf{e}, \mathbf{e}), \mathbf{f})$$

for unit tangent vectors \mathbf{e} and unit normal vectors \mathbf{f} on A . In a neighbourhood of each point on A , either G_1 has an absolute positive maximum or $G_1 \equiv 0$.

If G_1 has an absolute maximum, define $(\mathbf{e}_3, \mathbf{f}_4)$ to be where this maximum is attained. Set $\mathbf{f}_7 = \mathbf{f}_4 \times \mathbf{e}_3$, which is a unit normal vector orthogonal to \mathbf{f}_4 as A is associative. Since $(\mathbf{e}_3, \mathbf{f}_4)$ is a critical point of G_1 , we find that

$$g(\Pi_A(\mathbf{e}_3, \mathbf{e}), \mathbf{f}_4) = 0 \quad \text{and} \quad g(\Pi_A(\mathbf{e}_3, \mathbf{e}_3), \mathbf{f}) = 0 \quad (28)$$

for all tangents vectors \mathbf{e} and normal vectors \mathbf{f} such that $g(\mathbf{e}, \mathbf{e}_3) = g(\mathbf{f}, \mathbf{f}_4) = 0$.

Any unit tangent vector \mathbf{e}_1 orthogonal to \mathbf{e}_3 defines the remaining members of the framing for $T\mathcal{S}^7|_A$ using the associativity of A , since the final tangent direction is given by $\mathbf{e}_2 = \mathbf{e}_3 \times \mathbf{e}_1$ and the normal directions can be determined by $\mathbf{f}_5 = \mathbf{e}_1 \times \mathbf{f}_4$ and $\mathbf{f}_6 = \mathbf{f}_7 \times \mathbf{e}_1$. Thus, the remaining freedom we have is that of $\text{SO}(2)$ rotations in the plane bundle of vectors in TA orthogonal to \mathbf{e}_3 . Notice that these rotations will result in sympathetic rotations of the plane bundle of vectors in NA orthogonal to \mathbf{f}_4 and \mathbf{f}_7 . Thus, we can choose a unit tangent vector \mathbf{e}_1 , and hence complete the frame for $T\mathcal{S}^7|_A$, such that $h_{23}^5 + h_{31}^6 = 0$. Since $h_{31}^4 = h_{23}^4 = 0$ and $h_{33}^5 = h_{33}^6 = h_{33}^7 = 0$ by (28), we may set

$$\begin{aligned}r &= h_{31}^6, & s &= -h_{12}^7, & t &= \frac{1}{2}(h_{12}^4 + h_{11}^7), \\ u &= h_{12}^5, & v &= -h_{12}^6, & w &= \frac{1}{2}h_{31}^5\end{aligned}$$

and deduce the local formula (27) for Π_A using (25)-(26). If, in addition, $h_{31}^5 = 2w = 0$, we still have the freedom to rotate the plane field defined by \mathbf{e}_1 and \mathbf{e}_2 , so we can ensure that $h_{12}^6 = -v = 0$. Notice that r is assumed to be non-negative.

Now suppose that $G_1 \equiv 0$ on a neighbourhood of some point on A , so that $h_{jj}^a = 0$ for $j = 1, 2, 3$ and $a = 4, 5, 6, 7$ locally. Consider the function

$$G_2(\mathbf{e}, \mathbf{e}', \mathbf{f}) = g(\Pi_A(\mathbf{e}, \mathbf{e}'), \mathbf{f})$$

for orthogonal unit tangent vectors \mathbf{e} and \mathbf{e}' and unit normal vectors \mathbf{f} . This cannot be zero, unless A is totally geodesic, and so it has an absolute maximum at $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{f}_4)$, say. We can now define the remaining elements of our local frame by setting $\mathbf{e}_3 = \mathbf{e}_1 \times \mathbf{e}_2$ and using the previously mentioned formulae. The fact that $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{f}_4)$ is a critical point forces

$$g(\Pi_A(\mathbf{e}_3, \mathbf{e}), \mathbf{f}_4) = 0 \quad \text{and} \quad g(\Pi_A(\mathbf{e}_1, \mathbf{e}_2), \mathbf{f}) = 0 \quad (29)$$

for tangent vectors \mathbf{e} orthogonal to \mathbf{e}_3 and normal vectors \mathbf{f} orthogonal to \mathbf{f}_4 . We deduce from (25)-(26) and (29) that every term h_{jk}^a vanishes except $h_{12}^4 = h_{31}^5 = -h_{23}^6 = 2w$. \square

4.2 Reductions of the structure equations

We now observe that an associative 3-fold in \mathcal{S}^7 which arises from the geometry of \mathcal{S}^6 or \mathbb{C}^4 must have certain symmetries in its structure equations.

Note In this subsection, and in the remainder of the paper, we will use the notation of Propositions 4.3-4.4.

Example 4.7 (Products) Suppose that A is an associative 3-fold in \mathcal{S}^7 constructed from a pseudoholomorphic curve Σ in \mathcal{S}^6 as in Proposition 3.8(a). Then a frame for $T\mathcal{S}^7|_A$ may be chosen such that, for some function λ ,

$$\alpha_1 = \lambda\omega_2, \quad \alpha_2 = -\lambda\omega_1, \quad \text{and} \quad \beta_3^4 = \beta_3^5 = \beta_3^6 = \beta_3^7 = 0.$$

The latter condition is equivalent to $\Pi_A(\mathbf{e}_3, \cdot) = 0$ and implies, by (25)-(26), that $\beta_2^4 = \beta_1^7$, $\beta_2^5 = \beta_1^6$, $\beta_2^6 = -\beta_1^5$ and $\beta_2^7 = -\beta_1^4$. Here, $d\omega_3 = 0$, so that \mathbf{e}_3 defines the product direction orthogonal to Σ .

Example 4.8 (Lagrangians) Suppose that L is a Lagrangian submanifold in a totally geodesic \mathcal{S}^6 in \mathcal{S}^7 . Then L is associative by Proposition 3.8(b) and a

frame for $T\mathcal{S}^7|_L$ may be chosen such that

$$\beta_1^7 = \beta_2^7 = \beta_3^7 = 0 \quad \text{and} \quad \gamma = \alpha - \omega.$$

In this case, \mathbf{f}_7 is the direction orthogonal to the totally geodesic \mathcal{S}^6 containing L , and the equations (11)-(12) are equivalent to the statement that

$$\beta_L = \begin{pmatrix} -\beta_1^6 & -\beta_2^6 & -\beta_3^6 \\ \beta_1^5 & \beta_2^5 & \beta_3^5 \\ \beta_1^4 & \beta_2^4 & \beta_3^4 \end{pmatrix} \quad (30)$$

is a symmetric trace-free matrix of 1-forms.

Example 4.9 (Minimal Legendrians) Suppose that L is a minimal Legendrian submanifold of \mathcal{S}^7 . Then L is associative by Proposition 3.9 and there is a frame for $T\mathcal{S}^7|_L$ such that

$$\beta_1^7 = \beta_2^7 = \beta_3^7 = 0 \quad \text{and} \quad \gamma = \alpha + \omega.$$

If we define β_L as in (30), the conditions (11)-(12) correspond to β_L being symmetric and trace-free as in Example 4.8.

Example 4.10 (Links of complex cones) Let A be an associative 3-fold constructed from a holomorphic curve Σ in \mathbb{CP}^3 as in Proposition 3.10. Then there is a choice of frame for $T\mathcal{S}^7|_A$ such that

$$\alpha_1 = \omega_1, \quad \alpha_2 = \omega_2, \quad \text{and} \quad \beta_3^4 = \beta_3^5 = \beta_3^6 = \beta_3^7 = 0.$$

As in Example 4.7, we see that $\Pi_A(\mathbf{e}_3, \cdot) = 0$ and $\beta_2^4 = \beta_1^7$, $\beta_2^5 = \beta_1^6$, $\beta_2^6 = -\beta_1^5$ and $\beta_2^7 = -\beta_1^4$. Here, \mathbf{e}_3 defines the direction of the circle fibres of A over Σ .

5 Homogeneous examples

In this section we classify the associative 3-folds in \mathcal{S}^7 which arise as the orbits of closed 3-dimensional Lie subgroups G of $\text{Spin}(7)$. This is the analogue of the work in [36] on homogeneous (Lagrangian) links in \mathcal{S}^6 of coassociative cones.

The most obvious homogeneous associative 3-fold is a totally geodesic 3-sphere as in the following example.

Example 5.1 Let $A_0 \subseteq \mathcal{S}^7$ be given by

$$A_0 = \{(x_0, x_1, x_2, x_3, 0, 0, 0, 0) \in \mathbb{R}^8 : x_0^2 + x_1^2 + x_2^2 + x_3^2 = 1\}.$$

Then A_0 is an associative 3-sphere in \mathcal{S}^7 which is invariant under the action of $SU(2)^3/\mathbb{Z}_2$ by [29, Proposition 12.4.2]. Moreover, A_0 is totally geodesic, so has constant curvature 1.

We are lead to make the following convenient definition.

Definition 5.2 We say that an associative 3-fold in \mathcal{S}^7 is *simple* if it is a totally geodesic \mathcal{S}^3 .

By [29, Proposition 12.4.2] we have the following straightforward result.

Proposition 5.3 *Up to rigid motion, A_0 given in Example 5.1 is the unique simple associative 3-fold in \mathcal{S}^7 .*

Remark Proposition 5.3 shows that not every totally geodesic \mathcal{S}^3 in \mathcal{S}^7 is associative, since the family of all totally geodesic 3-spheres in \mathcal{S}^7 is 16-dimensional, whereas the associative subfamily is 12-dimensional.

The subgroup of $\text{Spin}(7)$ which fixes a real direction in \mathbb{R}^8 is isomorphic to G_2 . Therefore, if G acts trivially on an \mathbb{R} factor in \mathbb{R}^8 it arises as a subgroup of G_2 , and so associative G -orbits are Lagrangian in a totally geodesic \mathcal{S}^6 . These are then classified by [36] and will be discussed in §5.4, so we need only consider the case where G acts fully on \mathbb{R}^8 .

We clearly have a $U(1)^3$ subgroup of $\text{Spin}(7)$ which acts irreducibly on \mathbb{C}^4 , and this is the only $U(1)^3$ subgroup up to conjugation. Suppose now that G is a subgroup of $\text{Spin}(7)$ which is isomorphic to $SU(2)$ or $SO(3)$. We have an irreducible representation ρ_i of $SU(2)$ on \mathbb{R}^i for $i = 3, \dots, 8$. Therefore, the possible subgroups G can only have one of the following representations:

$$\rho_3 \oplus \rho_5; \quad \rho_4 \oplus \rho_4; \quad \text{and} \quad \rho_8.$$

Suppose G corresponds to the representation $\rho_3 \oplus \rho_5$. Since $G \subseteq \text{Spin}(7)$ and no element of $\text{Spin}(7)$ preserves a 3-dimensional subspace of \mathbb{R}^8 , we see that the action of G in this case must be reducible, leading to a contradiction. The second representation $\rho_4 \oplus \rho_4$ corresponds to the “diagonal” action of $SU(2)$ on $\mathbb{C}^2 \oplus \mathbb{C}^2 \cong \mathbb{C}^4$, and ρ_8 corresponds to the induced action of $SU(2)$ on $S^3\mathbb{C}^2 \cong \mathbb{C}^4$ from the standard action on \mathbb{C}^2 .

We therefore have the following subgroups to consider.

(i) $U(1)^3$ acting on \mathbb{C}^4 as

$$\begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} \mapsto \begin{pmatrix} e^{i\theta_1} z_1 \\ e^{i\theta_2} z_2 \\ e^{i\theta_3} z_3 \\ e^{i\theta_4} z_4 \end{pmatrix} \quad \begin{array}{l} \text{for } \theta_1, \theta_2, \theta_3, \theta_4 \in \mathbb{R} \\ \text{such that } \theta_1 + \theta_2 + \theta_3 + \theta_4 = 0. \end{array} \quad (31)$$

(ii) $SU(2)$ acting on \mathbb{C}^4 as

$$\begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} \mapsto \begin{pmatrix} az_1 + bz_2 \\ -\bar{b}z_1 + \bar{a}z_2 \\ az_3 + bz_4 \\ -\bar{b}z_3 + \bar{a}z_4 \end{pmatrix} \quad \begin{array}{l} \text{for } a, b \in \mathbb{C} \\ \text{such that } |a|^2 + |b|^2 = 1. \end{array} \quad (32)$$

(iii) $SU(2)$ acting on \mathbb{C}^4 as

$$\begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} \mapsto \begin{pmatrix} a^3 z_1 + \sqrt{3} a^2 b z_2 + \sqrt{3} a b^2 z_3 + b^3 z_4 \\ -\sqrt{3} a^2 \bar{b} z_1 + a(|a|^2 - 2|b|^2) z_2 + b(2|a|^2 - |b|^2) z_3 + \sqrt{3} \bar{a} b^2 z_4 \\ \sqrt{3} a \bar{b}^2 z_1 - \bar{b}(2|a|^2 - |b|^2) z_2 + \bar{a}(|a|^2 - 2|b|^2) z_3 + \sqrt{3} \bar{a}^2 b z_4 \\ -\bar{b}^3 z_1 + \sqrt{3} \bar{a} \bar{b}^2 z_2 - \sqrt{3} \bar{a}^2 \bar{b} z_3 + \bar{a}^3 z_4 \end{pmatrix} \quad (33)$$

for $a, b \in \mathbb{C}$ such that $|a|^2 + |b|^2 = 1$.

5.1 $U(1)^3$ orbits

Let A be an associative 3-fold which is an orbit of the action given in (31) and identify $\mathbb{R}^8 \cong \mathbb{C}^4$ as in Definition 2.1. By considering the Lie algebra associated with the group action (31), it is straightforward to see that the tangent space to the cone CA on A at a point $(z_1, z_2, z_3, z_4) \in A$ is spanned by the vectors:

$$\begin{aligned} X_0 &= (z_1, z_2, z_3, z_4), & X_1 &= (iz_1, 0, 0, -iz_4), \\ X_2 &= (0, iz_2, 0, -iz_4), & X_3 &= (0, 0, iz_3, -iz_4). \end{aligned}$$

Recall the decomposition (2). Clearly $\omega_0(X_j, X_k) = 0$ for $j, k \in \{1, 2, 3\}$ and so $\omega_0 \wedge \omega_0|_{CA} \equiv 0$. Thus CA is Cayley if and only if it is special Lagrangian in \mathbb{C}^4 by (2). However, the $U(1)^3$ -invariant special Lagrangian 4-folds in \mathbb{C}^4 are classified in [25, §III.3.A] and there is a unique (up to rigid motion) $U(1)^3$ -invariant, non-planar, special Lagrangian cone in \mathbb{C}^4 given by

$$\begin{aligned} \{(z_1, z_2, z_3, z_4) \in \mathbb{C}^4 : |z_1| = |z_2| = |z_3| = |z_4|, \\ \operatorname{Re}(z_1 z_2 z_3 z_4) = 0, \operatorname{Im}(z_1 z_2 z_3 z_4) > 0\}. \end{aligned}$$

This gives us the following example of an associative 3-fold in \mathcal{S}^7 .

Example 5.4 Let $A_1 \subseteq \mathcal{S}^7$ be given by:

$$\begin{aligned} A_1 &= \left\{ \frac{1}{2}(e^{i\theta_1}, e^{i\theta_2}, e^{i\theta_3}, e^{i\theta_4}) \in \mathbb{C}^4 : \theta_1, \theta_2, \theta_3, \theta_4 \in \mathbb{R} \right. \\ &\quad \left. \text{such that } \theta_1 + \theta_2 + \theta_3 + \theta_4 = \frac{\pi}{2} \right\}. \end{aligned}$$

Then A_1 is a minimal Legendrian, hence associative, 3-fold in \mathcal{S}^7 which is invariant under the action of $U(1)^3$ given in (31). Moreover, it is straightforward to see that A_1 is a 3-torus with constant curvature 0.

We deduce the following result.

Proposition 5.5 *Up to rigid motion, $A_1 \cong T^3$ given in Example 5.4 is the unique, connected, non-simple, $U(1)^3$ -invariant associative 3-fold in \mathcal{S}^7 , where the $U(1)^3$ action is given in (31).*

For possible interest we give the second structure equations for A_1 . We have that the matrices of 1-forms α, β, γ are given by:

$$\alpha = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}; \quad \beta = \begin{pmatrix} \omega_2 & \omega_1 & 0 \\ \omega_3 & 0 & \omega_1 \\ 0 & -\omega_3 & -\omega_2 \\ 0 & 0 & 0 \end{pmatrix}; \quad \gamma = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix}.$$

Thus, we have that $\gamma = \alpha + \omega$ and

$$d\omega_1 = d\omega_2 = d\omega_3 = 0.$$

Moreover, the second fundamental form is given by:

$$\Pi_A = 2\mathbf{f}_4 \otimes \omega_1\omega_2 + 2\mathbf{f}_5 \otimes \omega_3\omega_1 - 2\mathbf{f}_6 \otimes \omega_2\omega_3.$$

Remark A_1 is fibred by circles over the $U(1)^2$ -invariant minimal Legendrian 2-torus in a totally geodesic \mathcal{S}^5 in \mathcal{S}^7 .

5.2 $SU(2)$ orbits 1

Let A be an associative 3-fold in \mathcal{S}^7 invariant under the “diagonal” $SU(2)$ action given in (32), where we identify $\mathbb{C}^4 \cong \mathbb{R}^8$. The following matrices provide a basis for the Lie algebra associated with this action:

$$U_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}; \quad U_2 = \begin{pmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \end{pmatrix};$$

$$U_3 = \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix}.$$

Clearly, $[U_2, U_3] = 2U_1$, $[U_3, U_1] = 2U_2$ and $[U_1, U_2] = 2U_3$. The tangent space to A at a point (z_1, z_2, z_3, z_4) is thus spanned by the vectors

$$X_1 = (z_2, -z_1, z_4, -z_3), \quad X_2 = (iz_2, iz_1, iz_4, iz_3), \quad X_3 = (iz_1, -iz_2, iz_3, -iz_4).$$

By [36, Lemmas 5.3 & 5.6], using the $SU(2)$ action we can ensure that the metric g_A on A is given by $\lambda_1\omega_1^2 + \lambda_2\omega_2^2 + \lambda_3\omega_3^2$, where the ω_j form an orthonormal coframe for A and $\lambda_j = |X_j|^2$. However, $|X_j|^2 = 1$ for all j and so A must be totally geodesic. We deduce the following result.

Proposition 5.6 *Any connected associative orbit in \mathcal{S}^7 of the $SU(2)$ action given in (32) is simple.*

Note The non-singular Cayley 4-folds which are invariant under the $SU(2)$ action given in (32) were classified in [33, Theorem 5.3] (and, by other means, in [24]). One can use the proofs of these results to deduce Proposition 5.6.

5.3 $SU(2)$ orbits 2

Let A be an associative orbit of the action given in (33) and identify $\mathbb{R}^8 \cong \mathbb{C}^4$. The following matrices provide a basis for the associated Lie algebra to (33):

$$U_1 = \begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ -\sqrt{3} & 0 & 2 & 0 \\ 0 & -2 & 0 & \sqrt{3} \\ 0 & 0 & -\sqrt{3} & 0 \end{pmatrix}; \quad U_2 = \begin{pmatrix} 0 & \sqrt{3}i & 0 & 0 \\ \sqrt{3}i & 0 & 2i & 0 \\ 0 & 2i & 0 & \sqrt{3}i \\ 0 & 0 & \sqrt{3}i & 0 \end{pmatrix};$$

$$U_3 = \begin{pmatrix} 3i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -3i \end{pmatrix}.$$

Notice that $[U_2, U_3] = 2U_1$, $[U_3, U_1] = 2U_2$ and $[U_1, U_2] = 2U_3$.

We see immediately that the tangent space to A at (z_1, z_2, z_3, z_4) is spanned by the vectors:

$$X_1 = (\sqrt{3}z_2, -\sqrt{3}z_1 + 2z_3, -2z_2 + \sqrt{3}z_4, -\sqrt{3}z_3);$$

$$X_2 = (i\sqrt{3}z_2, i\sqrt{3}z_1 + 2iz_3, 2iz_2 + i\sqrt{3}z_4, i\sqrt{3}z_3);$$

$$X_3 = (3iz_1, iz_2, -iz_3, -3iz_4).$$

By [36, Lemmas 5.3 & 5.6], we can use the $SU(2)$ action to ensure that the metric g_A on A is given by $\lambda_1\omega_1^2 + \lambda_2\omega_2^2 + \lambda_3\omega_3^2$, where $\{\omega_1, \omega_2, \omega_3\}$ is an orthonormal

coframe for A and

$$\lambda_1 = |X_1|^2 = 4(|z_2|^2 + |z_3|^2) - 4\sqrt{3}\operatorname{Re}(z_1\bar{z}_3 + z_2\bar{z}_4) + 3; \quad (34)$$

$$\lambda_2 = |X_2|^2 = 4(|z_2|^2 + |z_3|^2) + 4\sqrt{3}\operatorname{Re}(z_1\bar{z}_3 + z_2\bar{z}_4) + 3; \quad (35)$$

$$\lambda_3 = |X_3|^2 = 8(|z_1|^2 + |z_4|^2) + 1, \quad (36)$$

using the fact that $|z_1|^2 + |z_2|^2 + |z_3|^2 + |z_4|^2 = 1$. The orthogonality of X_1 , X_2 and X_3 forces:

$$z_1\bar{z}_2 - z_3\bar{z}_4 = 0 \quad \text{and} \quad \operatorname{Im}(z_1\bar{z}_3 + z_2\bar{z}_4) = 0. \quad (37)$$

Let

$$\zeta_1 = 3|z_4|^2 + |z_3|^2 - |z_2|^2 - 3|z_1|^2; \quad (38)$$

$$\zeta_2 = 2\sqrt{3}z_1\bar{z}_2 + 4z_2\bar{z}_3 + 2\sqrt{3}z_3\bar{z}_4; \quad (39)$$

$$\zeta_3 = 36z_1^2z_4^2 - 12z_2^2z_3^2 + 16\sqrt{3}z_1z_3^3 + 16\sqrt{3}z_2^3z_4 - 72z_1z_2z_3z_4. \quad (40)$$

It is straightforward to see, using the equation (2) for Φ_0 , that, if CA is the cone on A , then

$$\Phi_0|_{CA} = \frac{|\zeta_1|^2 + |\zeta_2|^2 + \operatorname{Re}\zeta_3}{\sqrt{\lambda_1\lambda_2\lambda_3}} \operatorname{vol}_{CA}$$

at the point (z_1, z_2, z_3, z_4) . Thus CA is Cayley if and only if

$$\sqrt{\lambda_1\lambda_2\lambda_3} = |\zeta_1|^2 + |\zeta_2|^2 + \operatorname{Re}\zeta_3. \quad (41)$$

Recall that the Cayley condition on CA is also given by the vanishing of the 4-forms τ_j , $j = 1, \dots, 7$, given in Definition 3.4. The vanishing of these forms is equivalent to the following equations:

$$\operatorname{Im}\zeta_3 = 0; \quad (42)$$

$$8(z_1z_2 + \bar{z}_3\bar{z}_4)\zeta_1 + 4(\bar{z}_2\bar{z}_4 - z_1z_3)\zeta_2 + 4\sqrt{3}(z_1^2 - \bar{z}_4^2)\bar{\zeta}_2 = 0; \quad (43)$$

$$4(\sqrt{3}z_2^2 - \sqrt{3}\bar{z}_3^2 + z_1z_3 - \bar{z}_2\bar{z}_4)\zeta_1 - 4\sqrt{3}\operatorname{Re}(z_2z_3 + z_1z_4)\zeta_2 + 8(z_1z_2 + \bar{z}_3\bar{z}_4)\bar{\zeta}_2 = 0; \quad (44)$$

$$2(3z_2z_3 + 5\bar{z}_2\bar{z}_3 + 6z_1z_4 - 6\bar{z}_1\bar{z}_4)\zeta_1 + 4(\bar{z}_2^2 - \sqrt{3}z_2z_4)\zeta_2 - 4(\bar{z}_3^2 - \sqrt{3}z_1z_3)\bar{\zeta}_2 = 0. \quad (45)$$

We now make a quick observation.

Lemma 5.7 *Let A be an associative orbit through $(z_1, z_2, z_3, z_4) \in \mathbb{C}^4$ of the $\operatorname{SU}(2)$ action in (33), where we identify \mathbb{C}^4 with \mathbb{R}^8 as in Definition 2.1. Let $\zeta_1, \zeta_2, \zeta_3$ be given by (38)-(40).*

- (a) A is minimal Legendrian in \mathcal{S}^7 if and only if $\zeta_1 = \zeta_2 = \text{Im } \zeta_3 = 0$.
- (b) A is the link of a complex cone in \mathbb{C}^4 if and only if $\zeta_3 = 0$.

Proof: Let ω_0 and Ω_0 be the Kähler form and holomorphic volume form on \mathbb{C}^4 respectively. Then $\omega_0|_{CA} \equiv 0$ if and only if $\zeta_1 = \zeta_2 = 0$, and $\Omega_0|_{CA} = 0$ if and only if $\zeta_3 = 0$. The result follows. \square

As noted in [36, Remark 5.4], it is possible to permute the λ_j using the $\text{SU}(2)$ action but still preserve the orthogonality of the X_j . Thus the triple $(\lambda_1, \lambda_2, \lambda_3)$ is well-defined up to permutation and determines the metric on A . We now have the following lemma.

Lemma 5.8 *Let $A \subseteq \mathcal{S}^7$ be a connected, non-simple, associative orbit of the $\text{SU}(2)$ action given in (33). Up to permutation, $(\lambda_1, \lambda_2, \lambda_3)$ given by (34)-(36) is either $(3, 3, 9)$ or $(7, 7, 1)$. Moreover, the associative 3-folds with $(\lambda_1, \lambda_2, \lambda_3) = (3, 3, 9)$ are orbits through points $(\cos \theta, 0, 0, \sin \theta)$ for some $\theta \in [0, \frac{\pi}{4}]$, and if $(\lambda_1, \lambda_2, \lambda_3) = (7, 7, 1)$ then A is the orbit through $(0, \frac{1}{\sqrt{2}}, \frac{i}{\sqrt{2}}, 0)$.*

Proof: Suppose first, for a contradiction, that all of the λ_j are distinct. Since we are free to permute the λ_j we may assume that $\lambda_3 < \lambda_1 < \lambda_2$. From (34) and (35) we see that $\text{Re}(z_1 \bar{z}_3 + z_2 \bar{z}_4) > 0$, which means that at least one member of each of the pairs (z_1, z_4) and (z_2, z_3) is non-zero. Therefore, by (37), there must exist some real number μ such that

$$z_2 = \mu z_4 \quad \text{and} \quad z_3 = \mu z_1.$$

The fact that $\text{Re}(z_1 \bar{z}_3 + z_2 \bar{z}_4) > 0$ means that $\mu > 0$. Furthermore, $\lambda_3 < \lambda_1$ only if $\mu > \sqrt{3}$, using (34) and (36). We deduce from (43) that $\text{Im}(z_1 z_4) = 0$. Therefore the equations (42)-(45) are equivalent to the conditions:

$$\begin{aligned} \text{Im}(z_1^4 + z_4^4) &= 0; \\ \text{Im}(z_1 z_4) &= 0; \\ z_1^2 + \bar{z}_4^2 - \mu \sqrt{3}(\bar{z}_1^2 + z_4^2) &= 0; \\ \text{Im}((z_1^2 + \bar{z}_4^2)z_1 \bar{z}_4) &= 0. \end{aligned}$$

Since $\mu > \sqrt{3}$ and $(z_1, z_4) \neq (0, 0)$, we see that there are no solutions to these equations and we have reached our required contradiction.

Suppose now that at least two of the λ_j are equal. Then we may assume that $\lambda_1 = \lambda_2$ by our earlier remarks. Therefore from (34)-(35) we find that $\text{Re}(z_1 \bar{z}_3 + z_2 \bar{z}_4) = 0$. Combining this information with (37) we have that

$$z_1 \bar{z}_2 - z_3 \bar{z}_4 = 0 \quad \text{and} \quad z_1 \bar{z}_3 + z_2 \bar{z}_4 = 0.$$

Thus,

$$z_1 \bar{z}_2 \bar{z}_3 z_4 = |z_3|^2 |z_4|^2 = -|z_2|^2 |z_4|^2 = 0$$

and

$$\bar{z}_1 z_2 z_3 \bar{z}_4 = |z_1|^2 |z_2|^2 = -|z_1|^2 |z_3|^2 = 0.$$

We deduce that either $z_2 = z_3 = 0$ or $z_1 = z_4 = 0$. Thus the only possible triples $(\lambda_1, \lambda_2, \lambda_3)$ are $(3, 3, 9)$ and $(7, 7, 1)$.

For the first case, $z_2 = z_3 = 0$ and $\sqrt{\lambda_1 \lambda_2 \lambda_3} = 9$, so the Cayley condition (41) becomes:

$$9 = 9(|z_1|^2 - |z_4|^2)^2 + 36 \operatorname{Re}(z_1^2 z_4^2).$$

Since $|z_1|^2 + |z_4|^2 = 1$, we see that the Cayley condition is equivalent to $\operatorname{Im}(z_1 z_4) = 0$. Using the $\operatorname{SU}(2)$ action in (33) we can ensure that both z_1 and z_4 are real and non-negative, and that $z_1 \geq z_4$. Since $|z_1|^2 + |z_4|^2 = 1$, the result for $(\lambda_1, \lambda_2, \lambda_3) = (3, 3, 9)$ follows.

For the second case, $z_1 = z_4 = 0$ and $\sqrt{\lambda_1 \lambda_2 \lambda_3} = 7$, so we see from (41) and (42) that

$$7 = (|z_2|^2 - |z_3|^2)^2 + 16|z_2|^2 |z_3|^2 - 12 \operatorname{Re}(z_2^2 z_3^2) \quad \text{and} \quad \operatorname{Im}(z_2^2 z_3^2) = 0.$$

Therefore $\operatorname{Re}(z_2 z_3) = 0$ and $\operatorname{Im}(z_2 z_3) = \pm \frac{1}{2}$. Since $|z_2|^2 + |z_3|^2 = 1$ we deduce that $|z_2| = |z_3| = \frac{1}{\sqrt{2}}$ and $z_3 = \pm i z_2$. The result follows by using the $\operatorname{SU}(2)$ action in (33). \square

From this lemma we have the following examples of associative 3-folds in \mathcal{S}^7 .

Example 5.9 Let $A_2(\theta) \subseteq \mathcal{S}^7$ be the orbit of the $\operatorname{SU}(2)$ action (33) through $z_\theta = (\cos \theta, 0, 0, \sin \theta)$, for $\theta \in [0, \frac{\pi}{4}]$. Then $A_2(\theta)$ is associative and has an orthonormal coframe $\{\omega_1, \omega_2, \omega_3\}$ such that the induced metric on $A_2(\theta)$ is $3\omega_1^2 + 3\omega_2^2 + 9\omega_3^2$. One sees from the action (33) that z_θ has \mathbb{Z}_3 -stabilizer in $\operatorname{SU}(2)$. Therefore $A_2(\theta) \cong \operatorname{SU}(2)/\mathbb{Z}_3$.

Furthermore, by Lemma 5.7, $A_2(\theta)$ is the link of a complex cone if and only if $\theta = 0$ and it is minimal Legendrian if and only if $\theta = \frac{\pi}{4}$. However, rotation in the (z_1, z_4) -plane commutes with the $\operatorname{SU}(2)$ action in (33), so all the $A_2(\theta)$ are congruent up to rigid motion to $A_2 = A_2(0)$ which is $\operatorname{U}(2)$ -invariant.

Remark The fact that $A_2(0)$ and $A_2(\frac{\pi}{4})$ are congruent up to rigid motion is a special case of the main result in [3].

For possible interest we write down a form for the second structure equations describing $A_2(\theta)$ in the special cases $\theta = 0$ and $\theta = \frac{\pi}{4}$. Recall the matrices of

forms α, β, γ and the structure equations in Proposition 4.4. Here, for $\theta = 0$,

$$\alpha = \begin{pmatrix} \omega_1 \\ \omega_2 \\ -\frac{1}{3}\omega_3 \end{pmatrix}, \quad \beta = \frac{2\sqrt{3}}{3} \begin{pmatrix} 0 & 0 & 0 \\ \omega_2 & \omega_1 & 0 \\ \omega_1 & -\omega_2 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \gamma = \begin{pmatrix} 2\omega_1 \\ 2\omega_2 \\ \frac{2}{3}\omega_3 \end{pmatrix}.$$

Thus, we have that

$$d\omega_1 = \frac{2}{3}\omega_2 \wedge \omega_3, \quad d\omega_2 = \frac{2}{3}\omega_3 \wedge \omega_1, \quad d\omega_3 = 2\omega_1 \wedge \omega_2,$$

and the second fundamental form is given by:

$$\Pi_A = \frac{2\sqrt{3}}{3}(2\mathbf{f}_5 \otimes \omega_1\omega_2 + \mathbf{f}_6 \otimes (\omega_1^2 - \omega_2^2)).$$

We deduce that the components of the Riemann curvature tensor R_{ijkl} and normal curvature tensor R_{abkl}^\perp are determined by:

$$R_{2323} = 1, \quad R_{3131} = 1, \quad R_{1212} = -\frac{5}{3}, \quad R_{3112} = R_{1223} = R_{1332} = 0;$$

$$R_{65kl}^\perp = \frac{8}{3}(\delta_{k1}\delta_{l2} - \delta_{k2}\delta_{l1}), \quad R_{45kl}^\perp = R_{67kl}^\perp = R_{46kl}^\perp = R_{75kl}^\perp = R_{47kl}^\perp = 0.$$

If $\theta = \frac{\pi}{4}$, the second fundamental form, and thus β , remain the same, but

$$\alpha = (-\omega_1, -\omega_2, \frac{1}{3}\omega_3)^T \quad \text{and} \quad \gamma = (0, 0, \frac{4}{3}\omega_3)^T.$$

Thus, we deduce that

$$d\omega_1 = -\frac{2}{3}\omega_2 \wedge \omega_3, \quad d\omega_2 = -\frac{2}{3}\omega_3 \wedge \omega_1, \quad d\omega_3 = -2\omega_1 \wedge \omega_2,$$

The minimal Legendrian $A_2(\frac{\pi}{4})$ was first found in [35, §3.4].

Remarks Clearly $A_2(0)$ is the Hopf lift of the Veronese curve in \mathbb{CP}^3 , which is the degree 3 \mathbb{CP}^1 in \mathbb{CP}^3 given explicitly by

$$\{(x^3, \sqrt{3}x^2y, \sqrt{3}xy^2, y^3) \in \mathbb{CP}^3 : (x, y) \in \mathbb{CP}^1\}.$$

The minimal Legendrian $A_2(\frac{\pi}{4})$ is fibred by oriented geodesic circles over the Borůvka sphere in \mathcal{S}^4 , which is a 2-sphere with constant curvature $\frac{1}{3}$ (see [5]).

Example 5.10 Let $A_3 \subseteq \mathcal{S}^7$ be the orbit of the $\text{SU}(2)$ action (33) through $z = (0, \frac{1}{\sqrt{2}}, \frac{i}{\sqrt{2}}, 0)$. Then A_3 is associative and has an orthonormal coframe $\{\omega_1, \omega_2, \omega_3\}$ such that the metric on A_3 is $7\omega_1^2 + 7\omega_2^2 + \omega_3^2$. Since the point z has trivial stabilizer under the action (33), $A_3 \cong \text{SU}(2)$.

Furthermore, by Lemma 5.7 and the classification of homogeneous links of complex, coassociative and special Lagrangian cones, we see that A_3 does not arise from the complex, Lagrangian or minimal Legendrian geometries.

We now write down the second structure equations for A_3 . In this case,

$$\alpha = \frac{1}{7} \begin{pmatrix} \omega_1 \\ \omega_2 \\ 13\omega_3 \end{pmatrix}, \quad \beta = \frac{2\sqrt{3}}{7} \begin{pmatrix} -\omega_2 & -\omega_1 & 0 \\ -2\omega_3 & 0 & -2\omega_1 \\ 0 & 2\omega_3 & 2\omega_2 \\ \omega_1 & -\omega_2 & 0 \end{pmatrix}, \quad \gamma = \frac{1}{7} \begin{pmatrix} 0 \\ 0 \\ -36\omega_3 \end{pmatrix}.$$

Hence,

$$d\omega_1 = 2\omega_2 \wedge \omega_3, \quad d\omega_2 = 2\omega_3 \wedge \omega_1, \quad d\omega_3 = \frac{2}{7}\omega_1 \wedge \omega_2,$$

and the second fundamental form is given by:

$$\Pi_A = \frac{2\sqrt{3}}{7} (-2\mathbf{f}_4 \otimes \omega_1\omega_2 - 4\mathbf{f}_5 \otimes \omega_3\omega_1 + 4\mathbf{f}_6 \otimes \omega_2\omega_3 + \mathbf{f}_7 \otimes (\omega_1^2 - \omega_2^2)).$$

We deduce that the components of the Riemann curvature and normal curvature tensor are determined by:

$$\begin{aligned} R_{2323} &= \frac{1}{49}, \quad R_{3131} = \frac{1}{49}, \quad R_{1212} = \frac{25}{49}, \quad R_{3112} = R_{1223} = R_{2331} = 0; \\ R_{45kl}^\perp &= R_{67kl}^\perp = \frac{24}{49}(\delta_{k2}\delta_{l3} - \delta_{k3}\delta_{l2}), \quad R_{46kl}^\perp = R_{75kl}^\perp = \frac{24}{49}(\delta_{k3}\delta_{l1} - \delta_{k1}\delta_{l3}), \\ R_{65kl}^\perp &= 2R_{47kl}^\perp = \frac{48}{49}(\delta_{k1}\delta_{l2} - \delta_{k2}\delta_{l1}). \end{aligned}$$

We now conclude with the following result.

Proposition 5.11 *Let A be a connected, non-simple, associative 3-fold in \mathcal{S}^7 which is an orbit of the $\mathrm{SU}(2)$ action given in (33). Then, up to rigid motion, either $A = A_2$ as given in Example 5.9, or $A = A_3$ as given in Example 5.10.*

Proposition 5.11 gives an explicit example of an associative 3-fold in \mathcal{S}^7 which does not arise from other geometries and hence a new example of a Cayley cone.

Note Combining Propositions 5.5, 5.6 and 5.11 leads to Theorem 1.1.

5.4 Lagrangian orbits

For this subsection, let $\{\varepsilon_1, \dots, \varepsilon_7\}$ be an orthonormal basis for $\mathrm{Im} \mathbb{O}$ and identify $\mathrm{Im} \mathbb{O}$ with \mathbb{R}^7 . By Proposition 3.8(b), the associative 3-folds in \mathcal{S}^7 which lie

in a totally geodesic \mathcal{S}^6 are the Lagrangian submanifolds of the nearly Kähler 6-sphere. The Lagrangian submanifolds in \mathcal{S}^6 which arise as orbits of closed 3-dimensional Lie subgroups of G_2 were classified in [36]. We therefore have the following examples of associative 3-folds in \mathcal{S}^7 .

Example 5.12 Let $L_1 \subseteq \mathcal{S}^6$ be given by

$$L_1 = \left\{ \frac{\sqrt{5}}{3} \bar{q} \varepsilon_1 q + \frac{2}{3} q \varepsilon_5 : q \in \langle 1, \varepsilon_1, \varepsilon_2, \varepsilon_3 \rangle_{\mathbb{R}} \text{ with } |q| = 1 \right\}.$$

By [25, Theorem IV.3.2], L_1 is a Lagrangian $\mathrm{Sp}(1)$ -orbit in \mathcal{S}^6 . Moreover, L_1 is invariant under a $\mathrm{U}(2)$ subgroup of G_2 acting on $\mathrm{Im} \mathbb{O}$.

Example 5.13 Identify $\mathbb{R} \oplus \mathbb{C}^3 \cong \mathrm{Im} \mathbb{O}$ and let $L_2 \subseteq \mathcal{S}^6$ be given by

$$L_2 = \{ (0, z_1, z_2, z_3) \in \mathbb{R} \oplus \mathbb{C}^3 : |z_1|^2 + |z_2|^2 + |z_3|^2 = 1 \text{ and } z_1^2 + z_2^2 + z_3^2 = 0 \}.$$

Since L_2 is the link of a complex cone, it is Lagrangian in \mathcal{S}^6 . In fact, L_2 is the Hopf lift in \mathcal{S}^5 of the Veronese curve in \mathbb{CP}^2 , which is the degree 2 \mathbb{CP}^1 in \mathbb{CP}^2 . Moreover, L_2 is invariant under an $\mathrm{SO}(3)$ subgroup of G_2 .

Example 5.14 Identify $\mathrm{Im} \mathbb{O}$ with the homogeneous harmonic cubics $\mathcal{H}^3(\mathbb{R}^3)$ on \mathbb{R}^3 by:

$$\begin{aligned} \varepsilon_1 &\mapsto \frac{\sqrt{10}}{10} x(2x^2 - 3y^2 - 3z^2); \\ \varepsilon_2 &\mapsto -\sqrt{6}xyz; & \varepsilon_3 &\mapsto \frac{\sqrt{6}}{2} x(y^2 - z^2); \\ \varepsilon_4 &\mapsto -\frac{\sqrt{15}}{10} y(4x^2 - y^2 - z^2); & \varepsilon_5 &\mapsto -\frac{\sqrt{15}}{10} z(4x^2 - y^2 - z^2); \\ \varepsilon_6 &\mapsto \frac{1}{2} y(y^2 - 3z^2); & \varepsilon_7 &\mapsto -\frac{1}{2} z(z^2 - 3y^2). \end{aligned}$$

The standard $\mathrm{SO}(3)$ action on \mathbb{R}^3 induces an action on $\mathcal{H}^3(\mathbb{R}^3)$, hence on $\mathrm{Im} \mathbb{O}$.

Let L_3 be the orbit through ε_2 of this $\mathrm{SO}(3)$ action on $\mathrm{Im} \mathbb{O}$ and let L_4 be the orbit through ε_6 . By [36, Theorem 4.3] and observations in [34, Examples 6.6 & 6.15], $L_3 \cong \mathrm{SO}(3)/A_4$ and $L_4 \cong \mathrm{SO}(3)/S_3$ are Lagrangian. Moreover, L_3 has constant curvature $\frac{1}{16}$.

Remarks The cubic corresponding to ε_1 in Example 5.14 has $\mathrm{SO}(2)$ stabilizer under the $\mathrm{SO}(3)$ action. Therefore, the $\mathrm{SO}(3)$ -orbit through ε_1 is 2-dimensional. It is a linearly full pseudoholomorphic curve in \mathcal{S}^6 , as defined in Definition 2.8, known as the Borůvka sphere (see [5]) and has constant curvature $\frac{1}{6}$.

Note By [36, Theorem 4.4], the examples given in Examples 5.12-5.14 classify the non-totally geodesic homogeneous Lagrangians in \mathcal{S}^6 up to rigid motion. Combining this observation with Proposition 3.8(b) gives Theorem 1.2.

6 Curvature and rigidity

In this section we begin by classifying the associative 3-folds in \mathcal{S}^7 which have constant curvature. We then discuss further rigidity results for associative 3-folds primarily based on known facts from the study of minimal submanifolds.

6.1 Constant curvature examples

To motivate our study, we have a result on associative 3-folds lying in a totally geodesic \mathcal{S}^6 in \mathcal{S}^7 that follows from [16, Theorem 2] and [36, Lemma 2.5].

Proposition 6.1 *Let A be a connected associative 3-fold in \mathcal{S}^7 with constant curvature κ which lies in a totally geodesic \mathcal{S}^6 . Then either $\kappa = 1$ and A is simple, or $\kappa = \frac{1}{16}$ and $A = L_3$, given by Example 5.14, up to rigid motion.*

We now consider the general situation of an associative 3-fold A in \mathcal{S}^7 with constant curvature $\kappa = 1 - \rho^2$. We assume that $\rho > 0$ to avoid the simple case.

Recall the second structure equations (20)-(24) and the second fundamental form Π_A . We apply Proposition 4.6, which gives us local functions r, s, t, u, v, w on A defining Π_A . We see that the Gauss equation (22) is equivalent to the following conditions:

$$-r(3r + 2s) - 4w^2 = -\rho^2; \quad (46)$$

$$-r(3r - 2s) - 4w^2 = -\rho^2; \quad (47)$$

$$r^2 - 2s^2 - 2t^2 - 2u^2 - 2v^2 - 2w^2 = -\rho^2; \quad (48)$$

$$4uw = 4vw = -2r(t + 3w) = 0. \quad (49)$$

If $r \equiv 0$ then by (46), $w \neq 0$, so (49) implies that $u = v = 0$. By (47)-(48), $s^2 + t^2 = w^2$. It is straightforward to see that there are no solutions to (23)-(24) in this case unless $s \equiv 0$, so we may set $2t = 2w = \rho$. If $r \neq 0$, (49) forces $t + 3w = 0$ and (46)-(47) imply that $s \equiv 0$. If $w \equiv 0$ then by Proposition 4.6 we can set $v = 0$ and see from (46)-(48) that $u^2 = 2r^2$ and $3r^2 = \rho^2$. If, instead, $w \neq 0$, then (49) implies that $u = v = 0$ and (46)-(48) are equivalent to $r^2 = 4w^2$ and $4r^2 = \rho^2$.

We conclude that one of the following three cases occurs with curvature $\kappa = 1 - \rho^2$.

- (i) $\Pi_A = 2\rho\mathbf{f}_4 \otimes \omega_1\omega_2 + 2\rho\mathbf{f}_5 \otimes \omega_3\omega_1 - 2\rho\mathbf{f}_6 \otimes \omega_2\omega_3.$
- (ii) $\Pi_A = \frac{\sqrt{3}}{3}\rho\mathbf{f}_4 \otimes (2\omega_3^2 - \omega_1^2 - \omega_2^2) \pm \frac{\sqrt{3}}{3}\rho\mathbf{f}_5 \otimes (2\sqrt{2}\omega_1\omega_2 \mp 2\omega_2\omega_3) \pm \frac{\sqrt{3}}{3}\rho\mathbf{f}_6 \otimes (\sqrt{2}\omega_1^2 - \sqrt{2}\omega_2^2 \pm 2\omega_3\omega_1).$
- (iii) $\Pi_A = \frac{1}{2}\rho\mathbf{f}_4 \otimes (2\omega_3^2 - \omega_1^2 - \omega_2^2 \mp 2\omega_1\omega_2) \pm \rho\mathbf{f}_5 \otimes (\omega_3\omega_1 \mp \omega_2\omega_3) + \rho\mathbf{f}_6 \otimes (\omega_3\omega_1 \mp \omega_2\omega_3) \mp \rho\mathbf{f}_7 \otimes (\omega_1^2 - \omega_2^2).$

It is perhaps unsurprising that these three second fundamental forms turn out to be equivalent under transformations of the orthonormal frame for $T\mathcal{S}^7|_A$, so we need only study case (i). These considerations allow us to prove the following.

Proposition 6.2 *The only possible constant curvatures for an associative 3-fold in \mathcal{S}^7 are 0, $\frac{1}{16}$ and 1.*

Proof: Let A be a non-simple associative 3-fold with constant curvature $\kappa = 1 - \rho^2$. By our earlier observations we may assume that the second fundamental form Π_A of A is given locally by (i) above.

Recall the vectors of 1-forms ω , α and γ given in Propositions 4.3-4.4. There exist local functions a_{ij} and c_{ij} , for $1 \leq j, k \leq 3$, such that $\alpha_j = \sum_k a_{jk}\omega_k$ and $\gamma_j = \sum_k c_{jk}\omega_k$. It follows from (23) that $a_{jk} = c_{jk} = 0$ for $j \neq k$. Moreover, there is a local function λ such that $a_{jj} = \lambda$ and $c_{jj} = 1 - 7\lambda$ for all j . Thus, from (20), (22) and (24) we may deduce the following neat formulae:

$$d\omega_j \wedge \omega_j = 2\lambda\omega_1 \wedge \omega_2 \wedge \omega_3; \quad (50)$$

$$d(\lambda\omega_j) \wedge \omega_j = (1 + \lambda^2 - \rho^2)\omega_1 \wedge \omega_2 \wedge \omega_3; \quad (51)$$

$$d((1 - 7\lambda)\omega_j) \wedge \omega_j = ((1 - 7\lambda)^2 - \rho^2)\omega_1 \wedge \omega_2 \wedge \omega_3, \quad (52)$$

for all j . From the structure equations (20), (22) and (24) we see that λ must be constant. Hence, it follows from (50)-(52) that

$$\lambda^2 + \rho^2 = 1 \quad \text{and} \quad \lambda(16 - 63\lambda) + \rho^2 = 1.$$

We deduce that either $\lambda = 0$ or $\lambda = \frac{1}{4}$, so that $\kappa = \lambda^2$ is either 0 or $\frac{1}{16}$. \square

We then have the following observation.

Proposition 6.3 *For each $\kappa \in \{0, \frac{1}{16}, 1\}$ there is a unique connected associative 3-fold in \mathcal{S}^7 , up to rigid motion, which has constant curvature κ .*

Proof: The result is already known in the simple case $\kappa = 1$ by Proposition 5.3. For the other cases, since we may assume that the second fundamental

form is locally given as in (i) above, it is elementary to check that the exterior differential system on $\text{Spin}(7)$ corresponding to the structure equations in each case is involutive and its only nonzero Cartan character is $s_0 = 18$. Since $\dim \text{Spin}(7) = 21$, it follows that there is a unique associative 3-fold with a given constant curvature up to rigid motion. \square

Note Combining the results of Propositions 6.1-6.3 leads to Theorem 1.3.

6.2 Some rigidity results

We now present two results which follow from known theorems from the theory of minimal submanifolds in spheres.

Proposition 6.4 *Let A be an associative 3-fold in \mathcal{S}^7 , let S_A be its scalar curvature and let Π_A be its second fundamental form. If $S_A \geq 4$ or, equivalently, $|\Pi_A|^2 \leq 2$, then A is totally geodesic.*

Proof: This is immediate from Corollary 3.6 and [31, Theorems 3 & 3']. \square

Remarks A simple associative 3-fold has scalar curvature 6, so Proposition 6.4 states that the scalar curvature of an associative 3-fold cannot take values in $[4, 6)$. Given Theorem 1.3, it is natural to ask whether there are positive numbers ϵ and δ such that if either $|S_A| < \epsilon$ or $|S_A - \frac{3}{16}| < \delta$ then either A is flat or A has constant curvature $\frac{1}{16}$. There is also the broader question of whether the scalar curvature takes discrete values, which may be easier to answer in the associative case than for general minimal submanifolds in spheres.

Proposition 6.5 *Let A be an associative 3-fold in \mathcal{S}^7 and, for each $p \in A$, let $K_A(p)$ be the minimum of the sectional curvatures of A at p . If $K_A > \frac{5}{12}$ then A is totally geodesic.*

Proof: By Corollary 3.6 this is an application of the result of [30]. \square

Remark The result of [14] is that if $A \subseteq \mathcal{S}^6 \subseteq \mathcal{S}^7$ is associative (i.e. if A is Lagrangian in \mathcal{S}^6) and $K_A > \frac{1}{16}$ then A is totally geodesic. This suggests that there is possibly significant room for improvement in Proposition 6.5.

When trying to classify submanifolds, it is natural to consider the situation where the normal bundle of the submanifold is flat, or perhaps only “semi-flat”

when this notion is suitably defined. We now show that these conditions are very restrictive in the associative case. We begin with a convenient definition.

Definition 6.6 Let A be an associative 3-fold in \mathcal{S}^7 and recall the vectors ω , α and γ of 1-forms given in Propositions 4.3-4.4. Define tensors R_{ijkl}^\pm on A , using summation notation, by:

$$\begin{aligned} d[\alpha - \omega]_{ij} + [\alpha - \omega]_{im} \wedge [\alpha - \omega]_{mj} &= \frac{1}{2} R_{ijkl}^+ \omega_k \wedge \omega_l; \\ d[\gamma]_{ij} + [\gamma]_{im} \wedge [\gamma]_{mj} &= \frac{1}{2} R_{ijkl}^- \omega_k \wedge \omega_l, \end{aligned}$$

where $[\]$ is defined in (5).

Proposition 6.7 *An associative 3-fold in \mathcal{S}^7 has flat normal bundle if and only if it is totally geodesic.*

Proof: We use the notation of Proposition 4.4 and Definition 6.6, and the summation convention. By (24) the components of the curvature tensor R_{abkl}^\perp of the normal connection on an associative 3-fold A in \mathcal{S}^7 are determined by $\frac{1}{2} R_{abkl}^\perp \omega_k \wedge \omega_l = \beta_j^a \wedge \beta_j^b$, where $\beta_j^a = h_{jk}^a \omega_k$ by (21) and h_{jk}^a are the components of the second fundamental form Π_A as in Definition 4.5. Moreover, (24) allows us to relate R_{ijkl}^\pm to R_{abkl}^\perp as follows:

$$R_{23kl}^\pm = R_{54kl}^\perp \pm R_{76kl}^\perp, \quad R_{31kl}^\pm = R_{64kl}^\perp \pm R_{57kl}^\perp, \quad R_{12kl}^\pm = R_{56kl}^\perp \pm R_{47kl}^\perp.$$

Therefore, if $R_{abkl}^\perp = 0$ then $R_{ijkl}^\pm = 0$. However, we may quickly deduce from (22) that $R_{ijkl}^+ = h_{ik}^a h_{jl}^a - h_{il}^a h_{jk}^a$ so that $R_{ijij}^+ = -\frac{1}{2} \|\Pi_A\|^2$. Thus $R_{ijkl}^+ = 0$ implies that $\Pi_A = 0$, so A is totally geodesic. Conversely, if A is totally geodesic, $\Pi_A = \beta = 0$, so the normal bundle of A is flat by (24). \square

We may also improve the result further as follows.

Proposition 6.8 *In the notation of Definition 6.6, an associative 3-fold in \mathcal{S}^7 with $R_{ijkl}^- = 0$ is totally geodesic.*

Proof: Let A be an associative 3-fold in \mathcal{S}^7 with $R_{ijkl}^- = 0$. We choose a local frame for $T\mathcal{S}^7|_A$ as in Proposition 4.6. Then we may explicitly write down R_{ijkl}^- in terms of the local functions r, s, t, u, v, w as follows:

$$R_{2323}^- = R_{3131}^- = -3r^2 - 4tw, \tag{53}$$

$$R_{1212}^- = r^2 + 2s^2 + 2t^2 - 2u^2 - 2v^2 - 6w^2, \tag{54}$$

and

$$R_{3112}^- = 4tu + 4sv, \quad R_{1223}^- = -4su + 4tv, \quad (55)$$

$$R_{2113}^- = 4uw, \quad R_{3221}^- = 4vw, \quad (56)$$

$$R_{2331}^- = -4(2r - s)w, \quad R_{1332}^- = -4(2r + s)w. \quad (57)$$

Suppose, for a contradiction, that $r \neq 0$. Then at least one of $2r - s$ and $2r + s$ is non-zero, so $w = 0$ by (57). However, this forces $r = 0$ by (53), resulting in our required contradiction.

Suppose now that $r = 0$ but $w \neq 0$. Then by (53) we have that $t = 0$, and from (56)-(57) we deduce that $s = u = v = 0$. However, (54) then forces $w = 0$, again giving a contradiction.

Finally, if $r = w = 0$, then (55) implies that $s = t = 0$ or $u = v = 0$. In either case, (54) allows us to conclude that all the local functions r, s, t, u, v, w vanish identically. Thus $\Pi_A = 0$ and A is totally geodesic as claimed. \square

This result suggests that any naive notion of a “semi-flat” normal bundle for an associative 3-fold A in \mathcal{S}^7 would force A to be totally geodesic as it would naturally impose the condition $R_{ijkl}^- = 0$.

7 Ruled associative 3-folds

In this section we review the material we require from [19], which provides the classification of associative 3-folds in \mathcal{S}^7 that are *ruled* by oriented geodesic circles. The first key idea will be the relationship between ruled associative 3-folds and surfaces in the space of oriented geodesic circles in \mathcal{S}^7 which are *pseudo-holomorphic curves* with respect to a certain almost complex structure. The other major point is that there is a natural connection between these pseudo-holomorphic curves and *minimal surfaces* in \mathcal{S}^6 .

We begin by defining our objects of interest.

Definition 7.1 We say that a 3-dimensional submanifold A of \mathcal{S}^7 is *ruled* if there exists a smooth fibration $\pi : A \rightarrow \Gamma$ for some 2-manifold Γ whose fibres are oriented geodesic circles in \mathcal{S}^7 .

Remark The associative 3-folds given by Propositions 3.8(a) and 3.10 provide clear examples of ruled associative 3-folds.

Definition 7.2 Let \mathcal{C} denote the space of oriented geodesic circles in \mathcal{S}^7 . Notice that \mathcal{C} is naturally isomorphic to the Grassmannian of oriented 2-planes in \mathbb{R}^8 simply by sending an oriented 2-plane in \mathbb{R}^8 to its intersection with \mathcal{S}^7 .

By [19, Proposition 2.1], $\mathcal{C} \cong \text{Spin}(7)/\text{U}(3)$ and the $\text{U}(3)$ structure defines a $\text{Spin}(7)$ -invariant (non-integrable) almost complex structure $J_{\mathcal{C}}$ on \mathcal{C} . We call a surface Γ in \mathcal{C} a *pseudoholomorphic curve* if $J_{\mathcal{C}}(T_p\Gamma) = T_p\Gamma$ for all $p \in \Gamma$.

We can also view \mathcal{C} as the standard 6-quadric in \mathbb{CP}^7 by sending an oriented basis (\mathbf{a}, \mathbf{b}) for a geodesic circle in \mathcal{S}^7 to $[\frac{1}{2}(\mathbf{a} - i\mathbf{b})] \in \mathbb{CP}^7$. This identification induces the usual integrable complex structure $I_{\mathcal{C}}$ on \mathcal{C} .

Let $\mathbf{c} : \Gamma \rightarrow \mathcal{C}$ be a surface in \mathcal{C} . Then we may write

$$\mathbf{c}(p) = (\mathbf{a}(p), \mathbf{b}(p)) \quad (58)$$

where $\mathbf{a}, \mathbf{b} : \Gamma \rightarrow \mathcal{S}^7$ are everywhere orthogonal, and so define an oriented basis for a 2-plane in \mathbb{R}^8 for each $p \in \Gamma$. We may therefore associate to Γ a map $\mathbf{x} : \Gamma \times [0, 2\pi) \rightarrow \mathcal{S}^7$ given by

$$\mathbf{x}(p, q) = \mathbf{a}(p) \cos q + \mathbf{b}(p) \sin q. \quad (59)$$

Clearly, the image of \mathbf{x} is a ruled 3-dimensional submanifold of \mathcal{S}^7 . We can now characterise the associative condition for the image of \mathbf{x} in terms of the geometry of \mathcal{C} by [19, Proposition 2.1].

Proposition 7.3 *Use the notation of Definition 7.2.*

- (a) *A pseudoholomorphic curve $\mathbf{c} : \Gamma \rightarrow \mathcal{C}$ as given by (58) defines an associative 3-fold in \mathcal{S}^7 via $\mathbf{x} : \Gamma \times [0, 2\pi) \rightarrow \mathcal{S}^7$ as in (59).*
- (b) *Let A be a ruled associative 3-fold in \mathcal{S}^7 as in Definition 7.1. Then $\mathbf{c} : \Gamma \rightarrow \mathcal{C}$ given by $\mathbf{c}(p) = \pi^{-1}(p)$ is a pseudoholomorphic curve.*

We conclude from Proposition 7.3 that ruled associative 3-folds are in one-to-one correspondence with pseudoholomorphic curves in \mathcal{C} . Hence, they depend locally on 10 functions of 1 variable.

The next key observation is that we have a *fibration* of \mathcal{C} over the 6-sphere.

Definition 7.4 Let \mathcal{C} be as in Definition 7.2 and let $\mathbf{c} \in \mathcal{C}$. Then since $\mathcal{C} \cong \text{Spin}(7)/\text{U}(3)$, \mathbf{c} may be identified with a coset $g\text{U}(3)$ for some $g \in \text{Spin}(7)$, where $\text{U}(3)$ is the stabilizer of \mathbf{c} in $\text{Spin}(7)$. Now, there is a unique $\text{SU}(4) \subseteq \text{Spin}(7)$ containing the $\text{U}(3)$ stabilizer, so \mathbf{c} defines a coset $g\text{SU}(4)$ in $\text{Spin}(7)/\text{SU}(4)$. However, $\text{SU}(4) \cong \text{Spin}(6)$, so $\text{Spin}(7)/\text{SU}(4) \cong \mathcal{S}^6$. Thus, we have a \mathbb{CP}^3 fibration $\tau : \mathcal{C} \rightarrow \mathcal{S}^6$. Explicitly, if \mathbf{c} has (\mathbf{a}, \mathbf{b}) as an oriented orthonormal basis, then $\tau(\mathbf{c}) = \mathbf{a} \times \mathbf{b}$, where \times is the cross product on \mathbb{R}^8 .

Definition 7.4 describes a “twistor fibration” of \mathcal{C} over \mathcal{S}^6 in the sense of [38].

The fibration $\tau : \mathcal{C} \rightarrow \mathcal{S}^6$ allows us to relate pseudoholomorphic curves in \mathcal{C} to more familiar surfaces by [38, Theorem 3.5].

Proposition 7.5 *Use the notation of Definitions 7.2 and 7.4. Let Γ be a pseudoholomorphic curve in \mathcal{C} .*

- (a) *If $\tau(\Gamma) = p \in \mathcal{S}^6$, then Γ is a holomorphic curve in $\tau^{-1}(p) \cong \mathbb{CP}^3$.*
- (b) *If Γ does not lie in any fibre of τ , then $\tau(\Gamma)$ is a minimal surface in \mathcal{S}^6 .*

By Proposition 7.5, the general pseudoholomorphic curve in \mathcal{C} projects to a minimal surface in \mathcal{S}^6 under τ . Note that minimal surfaces in \mathcal{S}^6 depend locally on 8 functions of 1 variable, whereas pseudoholomorphic curves in \mathcal{C} depend on 10 functions of 1 variable locally. Therefore, we would expect every minimal surface in \mathcal{S}^6 to be the projection of a pseudoholomorphic curve in \mathcal{C} . Moreover, a general minimal surface should lift to a family of pseudoholomorphic curves in \mathcal{C} parameterised by 2 functions of 1 variable locally. By [19, Lemma 7.5] one has even more: namely, one can construct a family of pseudoholomorphic curves in \mathcal{C} from a minimal surface in \mathcal{S}^6 and *holomorphic data*. We now briefly describe the construction.

Example 7.6 (Ruled examples) Let $\mathbf{u} : \Sigma \rightarrow \mathcal{S}^6$ be a minimal surface. The normal bundle $N\Sigma$ of Σ in \mathcal{S}^6 has a spin structure so let $\text{Spin}(N\Sigma)$ be the principal spin bundle. This decomposes into positive and negative spin bundles, so we denote the associated spinor bundles over Σ by \mathbb{S}^+ and \mathbb{S}^- . Since Σ has a conformal structure, and the positive and negative spinors are interchanged by changing the orientation on Σ , we may focus on the positive spinors and define the bundle $W^+ = \mathbb{S}^+ \otimes T^{(1,0)}\Sigma$ over Σ . Let $\mathcal{X}(\Sigma) = \mathbb{P}(W^+)$, which is a holomorphic \mathbb{CP}^1 bundle over Σ .

Recall the notation of Definitions 7.2 and 7.4. It is observed in [19, §7] that $\mathcal{X}(\Sigma)$ is contained in $\mathbf{u}^*(\mathcal{C})$. Thus a surface Γ in $\mathcal{X}(\Sigma)$ defines a lift of Σ to \mathcal{C} ; i.e. a smooth map $c_\Gamma : \Sigma \rightarrow \mathcal{C}$ such that $\tau \circ c_\Gamma = \mathbf{u}$. By [19, Lemma 7.5], Γ is a holomorphic curve in $\mathcal{X}(\Sigma)$ if and only if the lift $c_\Gamma : \Sigma \rightarrow \mathcal{C}$ is a pseudoholomorphic curve. This pseudoholomorphic curve defines a ruled associative 3-fold in \mathcal{S}^7 by Proposition 7.3(a), which we denote by $A(\Sigma, \mathbf{u}, \Gamma)$.

Thus, given a minimal surface Σ in \mathcal{S}^6 , we have a family of ruled associative submanifolds of \mathcal{S}^7 locally parameterised by a holomorphic function from Σ to \mathbb{CP}^1 , as we expected from our earlier parameter count.

By [19, Theorem 7.8], we may now classify the ruled associative 3-folds in \mathcal{S}^7 as in Theorem 1.4, which we restate below.

Theorem 7.7 *Let A be a ruled associative 3-fold in \mathcal{S}^7 as in Definition 7.1. Then either*

- (a) A is given by Proposition 3.10, or
- (b) $A = A(\Sigma, \mathbf{u}, \Gamma)$ for some minimal surface $\mathbf{u} : \Sigma \rightarrow \mathcal{S}^6$ and holomorphic curve Γ in $\mathcal{X}(\Sigma)$ as in Example 7.6.

Note Theorem 7.7 is the natural extension of the main result in [34] for ruled Lagrangian submanifolds of \mathcal{S}^6 .

To conclude this section we make some further observations concerning ruled associative 3-folds in \mathcal{S}^7 , though we first give a definition for convenience.

Definition 7.8 By Proposition 7.3 there is an isomorphism ι between the set of ruled associative 3-folds in \mathcal{S}^7 and the set of pseudoholomorphic curves in \mathcal{C} , as given in Definition 7.2. Let $\varpi = \tau \circ \iota$, where τ is given in Definition 7.4.

Proposition 7.9 Let A be a ruled associative 3-fold in \mathcal{S}^7 and use the notation of Definition 7.8.

- (a) $\varpi(A)$ is a point if A is given by Proposition 3.10.
- (b) $\varpi(A)$ is a minimal surface in a totally geodesic \mathcal{S}^5 in \mathcal{S}^6 if A is minimal Legendrian in \mathcal{S}^7 .
- (c) $\varpi(A)$ is a pseudoholomorphic curve or a point in \mathcal{S}^6 if A is given by Proposition 3.8.

Proof: Recall the notation of Propositions 4.3-4.4, giving $\mathbf{x} : A \rightarrow \mathcal{S}^7$, \mathbf{e} and \mathbf{f} which define a moving G_2 frame for $T\mathcal{S}^7|_A$, and matrices of 1-forms α, β, γ over A . Further, let \mathbf{e}_3 define the ruling direction. If \times is the cross product on \mathcal{S}^7 given in Definition 2.4 we let

$$\begin{aligned} \mathbf{u} &= \mathbf{x} \times \mathbf{e}_3, & \mathbf{t}_1 &= \mathbf{x} \times \mathbf{e}_1, & \mathbf{t}_2 &= \mathbf{x} \times \mathbf{e}_2, \\ \mathbf{n}_1 &= \mathbf{x} \times \mathbf{f}_6, & \mathbf{n}_2 &= \mathbf{x} \times \mathbf{f}_5, & \mathbf{b}_1 &= \mathbf{x} \times \mathbf{f}_7, & \mathbf{b}_2 &= \mathbf{x} \times \mathbf{f}_4. \end{aligned}$$

From Definitions 7.4 and 7.8, we see that \mathbf{u} defines the immersion of $\varpi(A)$ in \mathcal{S}^6 . Moreover, using (17)-(19), the following structure equations hold on $\text{Spin}(7)$:

$$d\mathbf{u} = -\mathbf{t}_1(\alpha_2 - \omega_2) + \mathbf{t}_2(\alpha_1 - \omega_1) + \mathbf{n}_1\beta_3^6 + \mathbf{n}_2\beta_3^5 + \mathbf{b}_1\beta_3^7 + \mathbf{b}_2\beta_3^4; \quad (60)$$

$$d\mathbf{t}_1 = \mathbf{u}(\alpha_2 - \omega_2) - \mathbf{t}_2(\alpha_3 - \omega_3) + \mathbf{n}_1\beta_1^6 + \mathbf{n}_2\beta_1^5 + \mathbf{b}_1\beta_1^7 + \mathbf{b}_2\beta_1^4; \quad (61)$$

$$d\mathbf{t}_2 = -\mathbf{u}(\alpha_1 - \omega_1) + \mathbf{t}_1(\alpha_3 - \omega_3) + \mathbf{n}_1\beta_2^6 + \mathbf{n}_2\beta_2^5 + \mathbf{b}_1\beta_2^7 + \mathbf{b}_2\beta_2^4; \quad (62)$$

$$\begin{aligned} d\mathbf{n}_1 = & -\mathbf{u}\beta_3^6 - \mathbf{t}_1\beta_1^6 - \mathbf{t}_2\beta_2^6 + \frac{1}{2}\mathbf{n}_2(\alpha_3 + \omega_3 + \gamma_3) \\ & + \frac{1}{2}\mathbf{b}_1(\alpha_1 + \omega_1 - \gamma_1) - \frac{1}{2}\mathbf{b}_2(\alpha_2 + \omega_2 + \gamma_2); \end{aligned} \quad (63)$$

$$\begin{aligned} d\mathbf{n}_2 = & -\mathbf{u}\beta_3^5 - \mathbf{t}_1\beta_1^5 - \mathbf{t}_2\beta_2^5 - \frac{1}{2}\mathbf{n}_1(\alpha_3 + \omega_3 + \gamma_3) \\ & - \frac{1}{2}\mathbf{b}_1(\alpha_2 + \omega_2 - \gamma_2) - \frac{1}{2}\mathbf{b}_2(\alpha_1 + \omega_1 + \gamma_1); \end{aligned} \quad (64)$$

$$\begin{aligned} d\mathbf{b}_1 = & -\mathbf{u}\beta_3^7 - \mathbf{t}_1\beta_1^7 - \mathbf{t}_2\beta_2^7 - \frac{1}{2}\mathbf{n}_1(\alpha_1 + \omega_1 - \gamma_1) + \frac{1}{2}\mathbf{n}_2(\alpha_2 + \omega_2 - \gamma_2) \\ & + \frac{1}{2}\mathbf{b}_2(\alpha_3 + \omega_3 - \gamma_3); \end{aligned} \quad (65)$$

$$\begin{aligned} d\mathbf{b}_2 = & -\mathbf{u}\beta_3^4 - \mathbf{t}_1\beta_1^4 - \mathbf{t}_2\beta_2^4 + \frac{1}{2}\mathbf{n}_1(\alpha_2 + \omega_2 + \gamma_2) + \frac{1}{2}\mathbf{n}_2(\alpha_1 + \omega_1 + \gamma_1) \\ & - \frac{1}{2}\mathbf{b}_1(\alpha_3 + \omega_3 - \gamma_3). \end{aligned} \quad (66)$$

We immediately deduce from (60)-(66) that the image of \mathbf{u} is a point in \mathcal{S}^6 if and only if $\alpha_1 = \omega_1$, $\alpha_2 = \omega_2$ and $\beta_3^4 = \beta_3^5 = \beta_3^6 = \beta_3^7 = 0$. From Example 4.10 we see that these conditions are satisfied by the Hopf lift of a holomorphic curve in \mathbb{CP}^3 , from which (a) follows immediately.

Otherwise, (60)-(66) are the structure equations for a surface in \mathcal{S}^6 , where $\mathbf{t}_1, \mathbf{t}_2$ are orthogonal unit tangent vectors and $\mathbf{n}_1, \mathbf{n}_2, \mathbf{b}_1, \mathbf{b}_2$ are orthogonal unit normal vectors. Furthermore, $\Omega_1 = -\alpha_2 + \omega_2$ and $\Omega_2 = \alpha_1 - \omega_1$ define an orthonormal coframe for $\varpi(A)$. Recall Proposition 4.6. Since A is ruled and \mathbf{e}_3 is the ruling direction, the local function r in the formula for the second fundamental form Π_A given by Proposition 4.6 must vanish. Thus we may choose \mathbf{e} and \mathbf{f} such that Π_A has the following local form:

$$\begin{aligned} \Pi_A = & \mathbf{f}_4 \otimes (s\omega_1^2 - s\omega_2^2 + 2(t+w)\omega_1\omega_2) \\ & + \mathbf{f}_5 \otimes (v\omega_1^2 - v\omega_2^2 + 2u\omega_1\omega_2 + 4w\omega_3\omega_1) \\ & + \mathbf{f}_6 \otimes (u\omega_1^2 - u\omega_2^2 - 2v\omega_1\omega_2 - 4w\omega_2\omega_3) \\ & + \mathbf{f}_7 \otimes ((t-w)\omega_1^2 - (t-w)\omega_2^2 - 2s\omega_1\omega_2). \end{aligned} \quad (67)$$

Using (61), (62) and (67) we may write down the second fundamental form of $\varpi(A)$ in terms of the normal vectors and orthonormal coframe for $\varpi(A)$. It is straightforward to check that $\varpi(A)$ is minimal, thus verifying Theorem 7.7(b).

We observe that $d\mathbf{b}_1 = 0$, so $\varpi(A)$ is contained in a totally geodesic \mathcal{S}^5 , if and only if $\beta_1^7 = \beta_2^7 = \beta_3^7 = 0$ and $\gamma = \alpha + \omega$. Part (b) follows from the observations in Example 4.9.

For part (c), first notice that if we take a pseudoholomorphic curve Σ in \mathcal{S}^6 and construct an associative 3-fold A from it as in Proposition 3.8(a), then $\varpi(A) \cong \Sigma$. By Proposition 3.8(b), an associative 3-fold A lying in a totally geodesic \mathcal{S}^6 is Lagrangian, so it is ruled over a pseudoholomorphic curve in \mathcal{S}^6 or is the Hopf lift of a holomorphic curve in \mathbb{CP}^2 by [34, Theorem 7.5]. Thus $\varpi(A)$ is either a pseudoholomorphic curve or a point as claimed. \square

Remarks

- (a) Though A_1 and A_3 given in Examples 5.4 and 5.10 are fibered by circles over a surface, they are not ruled as the circles are not geodesics in \mathcal{S}^7 . The same is true for L_1 and L_3 given in Examples 5.12 and 5.14.
- (b) The associative 3-folds $A_2(\theta)$, L_2 and L_4 given in Examples 5.9, 5.13 and 5.14 are ruled. Moreover, $\varpi(A_2(0))$ and $\varpi(L_2)$ are each a point, $\varpi(A_2(\frac{\pi}{4}))$ is the Borůvka sphere $\mathcal{S}^2(\frac{1}{3})$ in a totally geodesic \mathcal{S}^4 in \mathcal{S}^6 , and $\varpi(L_4)$ is the linearly full Borůvka sphere $\mathcal{S}^2(\frac{1}{6})$ in \mathcal{S}^6 .

8 Chen's equality

In the study of submanifolds of manifolds with constant curvature, it has been fruitful to analyse those submanifolds satisfying the curvature condition known as *Chen's equality*, particularly in the case when the submanifold is minimal. In this section we classify the associative 3-folds in \mathcal{S}^7 satisfying Chen's equality.

We begin by defining the curvature condition we wish to study.

Definition 8.1 Let A be an associative 3-fold in \mathcal{S}^7 . Let S_A be the scalar curvature of A and let $K_A(p)$ be the minimum of the sectional curvatures of A at p . Since A is minimal by Corollary 3.6, it follows from [12, Lemma 3.2] that

$$\delta_A = \frac{1}{2}S_A - K_A \leq 2.$$

The curvature condition $\delta_A = 2$, introduced in [12], is known as *Chen's equality*. By [12, Lemma 3.2], Chen's equality is equivalent to the existence of a non-zero tangent vector \mathbf{v} on A at each point such that $\Pi_A(\mathbf{v}, \cdot) = 0$, where Π_A is the second fundamental form of A .

The Lagrangian submanifolds of \mathcal{S}^6 which satisfy Chen's equality are classified in [15]. These submanifolds are necessarily associative when embedded in \mathcal{S}^7 by Proposition 3.8(b). We now give some further examples of associative 3-folds in \mathcal{S}^7 which satisfy Chen's equality.

Proposition 8.2 *Associative 3-folds given by Propositions 3.8(a) and 3.10 satisfy Chen's equality.*

Proof: The result follows immediately from the observations in Examples 4.7 and 4.10 and Definition 8.1. \square

The examples of associative 3-folds in \mathcal{S}^7 satisfying Chen's equality given by Proposition 8.2 depend locally on 4 functions of 1 variable. However, it is a straightforward calculation using exterior differential systems to see that associative 3-folds satisfying Chen's equality depend on 6 functions of 1 variable. We begin our analysis of these general examples with the following key result.

Lemma 8.3 *An associative 3-fold in \mathcal{S}^7 which satisfies Chen's equality is ruled.*

Proof: Let Π_A be the second fundamental form of an associative 3-fold $A \subseteq \mathcal{S}^7$ satisfying Chen's equality and recall the notation of Propositions 4.3-4.4. By Definition 8.1 there exists a local unit tangent vector field \mathbf{e}_3 on A such that $\Pi_A(\mathbf{e}_3, \cdot) = 0$. We thus have that Π_A must locally be of the form:

$$\begin{aligned} \Pi_A = & \mathbf{f}_4 \otimes (s(\omega_1^2 - \omega_2^2) + 2t\omega_1\omega_2) + \mathbf{f}_5 \otimes (v(\omega_1^2 - \omega_2^2) + 2u\omega_1\omega_2) \\ & + \mathbf{f}_6 \otimes (u(\omega_1^2 - \omega_2^2) - 2v\omega_1\omega_2) + \mathbf{f}_7 \otimes (t(\omega_1^2 - \omega_2^2) - 2s\omega_1\omega_2). \end{aligned} \quad (68)$$

From (23) we quickly deduce that the connection forms α_1 and α_2 on A must be linear combinations of ω_1 and ω_2 . Thus, it follows from the first structure equations (17) and (18) given in Proposition 4.3 that

$$d\mathbf{x} = \mathbf{e}_3\omega_3, \quad d\mathbf{e}_1 = 0, \quad d\mathbf{e}_2 = 0 \quad \text{and} \quad d\mathbf{e}_3 = -\mathbf{x}\omega_3 \quad \text{modulo } \omega_1, \omega_2.$$

These equations define circles in A which are geodesics in \mathcal{S}^7 . Thus, A is ruled and \mathbf{e}_3 defines the direction of the ruling. \square

Recall that the ruled associative 3-folds depend locally on 10 functions of 1 variable. Thus, the associative 3-folds satisfying Chen's equality are a distinguished subfamily of the ruled family. By Theorem 7.7, to identify this subfamily we need to understand Chen's equality as a condition on minimal surfaces in \mathcal{S}^6 and their lifts to pseudoholomorphic curves in the space \mathcal{C} of oriented geodesic circles in \mathcal{S}^7 .

We begin by identifying the distinguished minimal surfaces in \mathcal{S}^6 we seek, for which we require a definition.

Definition 8.4 Let Σ be a surface in \mathcal{S}^6 and let Π_Σ be its second fundamental form. The *first ellipse of curvature* at $p \in \Sigma$ is given by

$$\{\Pi_\Sigma(\mathbf{v}, \mathbf{v}) : \mathbf{v} \in T_p\Sigma, |\mathbf{v}| = 1\} \subseteq N_p\Sigma.$$

This is an ellipse in the normal space $N_p\Sigma$ whenever it is non-degenerate.

Following [37], we say that Σ is *isotropic* if its first ellipse of curvature is a (possibly degenerate) circle at each point.

Remarks Recall that if Σ is a surface in \mathcal{S}^6 , we may define the third fundamental form $\text{III}_\Sigma \in C^\infty(S^3 T^* \Sigma; N\Sigma)$ by setting $\text{III}_\Sigma(X, Y, Z)$ to be the component of $\nabla_X \Pi_\Sigma(Y, Z)$ which is orthogonal to $T\Sigma$ and the image of Π_Σ , where ∇ is the Levi-Civita connection on \mathcal{S}^6 . The second ellipse of curvature at $p \in \Sigma$ is given by

$$\{\text{III}_\Sigma(\mathbf{v}, \mathbf{v}) : \mathbf{v} \in T_p \Sigma, |\mathbf{v}| = 1\} \subseteq N_p \Sigma.$$

A minimal surface whose first and second ellipses of curvature are all circles or points is called *superminimal*.

Proposition 8.5 *Let A be an associative 3-fold in \mathcal{S}^7 which satisfies Chen's equality and use the notation of Definitions 7.8 and 8.4. Then $\varpi(A) \subseteq \mathcal{S}^6$ is either a point or an isotropic minimal surface.*

Proof: Since A satisfies Chen's equality its second fundamental form can be written locally as in (68). However, we still have freedom to transform the orthonormal frame for $T\mathcal{S}^7|_A$ to set $s = t = v = 0$. Thus, locally,

$$\Pi_A = 2u\mathbf{f}_5 \otimes \omega_1 \omega_2 + u\mathbf{f}_6 \otimes (\omega_1^2 - \omega_2^2). \quad (69)$$

We need only consider the case where $\varpi(A) = \Sigma$ is a surface by Propositions 7.9(a) and 8.2.

Recall the proof of Proposition 7.9. From the Codazzi equation (23) and the structure equations (61)-(62) for Σ , we calculate the second fundamental form of Σ to be given locally by:

$$\Pi_\Sigma = \mathbf{n}_1 \otimes (U(\Omega_1^2 - \Omega_2^2) - 2V\Omega_1\Omega_2) + \mathbf{n}_2 \otimes (V(\Omega_1^2 - \Omega_2^2) + 2U\Omega_1\Omega_2),$$

where $\Omega_1 = -\alpha_2 + \omega_2$, $\Omega_2 = \alpha_1 - \omega_1$ and U, V are multiples of u determined by α_1 and α_2 . Thus the first ellipse of curvature of Σ at a point p is given by:

$$\{\cos q(U(p)\mathbf{n}_1(p) + V(p)\mathbf{n}_2(p)) + \sin q(U(p)\mathbf{n}_2(p) - V(p)\mathbf{n}_1(p)) : q \in [0, 2\pi)\}.$$

Since this is clearly a circle or a point for each p the result follows. \square

Remark A_2 given by Example 5.9 satisfies Chen's equality.

From Propositions 7.5 and 8.5 we see that the pseudoholomorphic curves in \mathcal{C} corresponding to the associative 3-folds in \mathcal{S}^7 satisfying Chen's equality must either lie in a fibre of the fibration $\tau : \mathcal{C} \rightarrow \mathcal{S}^6$ given in Definition 7.4 or project to an isotropic minimal surface under τ . In fact, we can describe these

pseudoholomorphic curves using the fibration τ , but first we must study the structure equations for a pseudoholomorphic curve in \mathcal{C} .

Given a pseudoholomorphic curve $\mathbf{c} : \Gamma \rightarrow \mathcal{C}$, we can choose orthogonal $\mathbf{a}, \mathbf{b} : \Gamma \rightarrow \mathcal{S}^7$ such that $\mathbf{c}(p)$ has $(\mathbf{a}(p), \mathbf{b}(p))$ as an oriented orthonormal basis for each $p \in \Gamma$. Define $\mathbf{v}_0 = \frac{1}{2}(\mathbf{a} - i\mathbf{b})$ and let $\mathbf{v} = (\mathbf{v}_1 \ \mathbf{v}_2 \ \mathbf{v}_3)$ be a unitary frame for $T^{(1,0)}\mathcal{C}|_\Gamma$ adapted such that \mathbf{v}_1 spans $T^{(1,0)}\Gamma$, so that $\mathbf{v}_2, \mathbf{v}_3$ are normal to Γ in \mathcal{C} . Finally, we can form $\bar{\mathbf{h}} = (\mathbf{v}_0 \ \mathbf{v}_1 \ \mathbf{v}_2 \ \mathbf{v}_3 \ \bar{\mathbf{v}}_0 \ \bar{\mathbf{v}}_1 \ \bar{\mathbf{v}}_2 \ \bar{\mathbf{v}}_3)$.

Since we can view $\text{Spin}(7)$ as a $\text{U}(3)$ bundle over \mathcal{C} , $\mathbf{h}^{-1}d\mathbf{h} = \psi$ takes values in $\mathfrak{spin}(7)$ as given in Proposition 4.1 and may be written as

$$\psi = \begin{pmatrix} i\rho & -\bar{\mathbf{h}}^T & 0 & -\theta^T \\ \mathbf{h} & \kappa & \theta & [\bar{\theta}] \\ 0 & -\bar{\theta}^T & -i\rho & -\mathbf{h}^T \\ \bar{\theta} & [\theta] & \bar{\mathbf{h}} & \bar{\kappa} \end{pmatrix},$$

where \mathbf{h}, ρ, θ and κ are 1-forms on $\text{Spin}(7)$ taking values in the appropriate spaces of matrices as described in Proposition 4.1 and $[\]$ is defined in (5). Thus, $d\mathbf{h} = \mathbf{h}\psi$ and $d\psi + \psi \wedge \psi = 0$ are the structure equations for the adapted frame bundle over Γ . Moreover, the complex-valued 1-forms given by \mathbf{h} and θ define the almost complex structure $J_{\mathcal{C}}$ on \mathcal{C} by declaring a 1-form on \mathcal{C} to be a $(1, 0)$ -form if its pullback to $\text{Spin}(7)$ lies in the subbundle defined by \mathbf{h}_j and θ_j . From these considerations we can immediately deduce the following result.

Proposition 8.6 *Recall Definition 7.2 and let Γ be a pseudoholomorphic curve in \mathcal{C} . Using the notation above, the first structure equations for the adapted frame bundle over Γ are given by:*

$$d\mathbf{v}_0 = i\mathbf{v}_0\rho + \mathbf{v}\mathbf{h} + \bar{\mathbf{v}}\bar{\theta}; \quad (70)$$

$$d\mathbf{v} = -\mathbf{v}_0\bar{\mathbf{h}}^T + \mathbf{v}\kappa - \bar{\mathbf{v}}_0\bar{\theta}^T + \bar{\mathbf{v}}[\theta], \quad (71)$$

where $[\]$ is defined in (5). The second structure equations for the adapted frame bundle over Γ are given by:

$$d\mathbf{h} = -(\kappa - i\rho \text{Id}) \wedge \mathbf{h}; \quad (72)$$

$$d\theta = -(\kappa + i\rho \text{Id}) \wedge \theta; \quad (73)$$

$$d\kappa = -\kappa \wedge \kappa + \mathbf{h} \wedge \bar{\mathbf{h}}^T + \theta \wedge \bar{\theta}^T - [\bar{\theta}] \wedge [\theta], \quad (74)$$

where Id is the 3×3 identity matrix.

We see from (72)-(73) that the vectors of forms \mathbf{h} and θ push down to the pseudoholomorphic curve Γ . It is observed in [19, §5] that \mathbf{h} and θ are related to the fibration τ given in Definition 7.4 in the following manner.

Definition 8.7 Use the notation of Definitions 7.2 and 7.4 and Proposition 8.6 and consider \mathcal{C} with its almost complex structure $J_{\mathcal{C}}$. Let \mathcal{H} and \mathcal{V} be the subbundles of $T^{(1,0)}\mathcal{C}$ spanned by the vectors on which \mathfrak{h} and θ vanish respectively. Then $T^{(1,0)}\mathcal{C} = \mathcal{H} \oplus \mathcal{V}$ and, by [19, Lemma 5.1], $\tau_* : \mathcal{H} \rightarrow T^{(1,0)}\mathcal{S}^6$ is an isomorphism, where we consider \mathcal{S}^6 with the almost complex structure described in Definition 2.8. Thus, \mathcal{H} and \mathcal{V} can be viewed as horizontal and vertical distributions with respect to the fibration $\tau : \mathcal{C} \rightarrow \mathcal{S}^6$.

A pseudoholomorphic curve Γ in \mathcal{C} is *vertical* if $\theta|_{\Gamma} = 0$. By [19, Lemma 5.1 & Corollary 5.3], Γ is vertical if and only if it lies in a single fibre of τ , and may be viewed as a holomorphic curve in \mathbb{CP}^3 .

A pseudoholomorphic curve Γ in \mathcal{C} is *horizontal* if $\mathfrak{h}|_{\Gamma} = 0$. By [19, Corollary 5.3], a horizontal pseudoholomorphic curve is a holomorphic curve with respect to the complex structure $I_{\mathcal{C}}$ on \mathcal{C} , and is algebraic in the 6-quadric in \mathbb{CP}^7 .

We now define a new family of pseudoholomorphic curves in \mathcal{C} .

Definition 8.8 Recall the notation of Definition 7.2 and Proposition 8.6. Let $\lambda = \mathfrak{h} \times \theta$; i.e.

$$\lambda_1 = \mathfrak{h}_2 \circ \theta_3 - \mathfrak{h}_3 \circ \theta_2, \quad \lambda_2 = \mathfrak{h}_3 \circ \theta_1 - \mathfrak{h}_1 \circ \theta_3, \quad \lambda_3 = \mathfrak{h}_1 \circ \theta_2 - \mathfrak{h}_2 \circ \theta_1.$$

By (72)-(73), $d\lambda = -2\kappa \wedge \lambda$, so λ pushes down to pseudoholomorphic curves in \mathcal{C} . We say that a pseudoholomorphic curve Γ in \mathcal{C} is *linear* if $\lambda|_{\Gamma} = 0$.

Clearly, horizontal and vertical pseudoholomorphic curves in \mathcal{C} are linear. Using exterior differential systems, one sees that the linear pseudoholomorphic curves depend locally on 6 functions of 1 variable, whereas the horizontal and vertical curves depend locally on 4 functions of 1 variable.

The utility of Definition 8.8 becomes apparent in our next result.

Proposition 8.9 *Recall the notation of Definitions 7.2 and 8.8. Associative 3-folds in \mathcal{S}^7 satisfying Chen's equality are in one-to-one correspondence with linear pseudoholomorphic curves in \mathcal{C} .*

Proof: Recall the notation of Propositions 4.3-4.4 and 8.6.

Let A be an associative submanifold of \mathcal{S}^7 satisfying Chen's equality. One can adapt frames over A so that the second fundamental form Π_A of A has the local form as in (69). Since A is ruled by Lemma 8.3, A defines a pseudoholomorphic curve Γ in \mathcal{C} as in Proposition 7.3(b). From (6)-(7) and (69) we see that the vectors of 1-forms \mathfrak{h} and θ appearing in the structure equations (70)-(74) in Proposition 8.6 satisfy $\mathfrak{h} = (\mathfrak{h}_1, 0, 0)$ and $\theta = (\theta_1, 0, 0)$. Thus Γ is linear.

Suppose now that Γ is a linear pseudoholomorphic curve in \mathcal{C} and let A be the ruled associative 3-fold constructed as in Proposition 7.3(a). We see from (6)-(7), since $\eta|_A = 0$, that we have the freedom to adapt frames on Γ such that $\mathfrak{h}_2 + \theta_2 = \mathfrak{h}_3 + \theta_3 = 0$. Thus the condition $\lambda|_\Gamma = 0$ implies that

$$\mathfrak{h}_j \circ \theta_1 - \theta_j \circ \mathfrak{h}_1 = \mathfrak{h}_j \circ (\mathfrak{h}_1 + \theta_1) = 0$$

on Γ for $j = 2, 3$. From (6)-(7) we see that $\mathfrak{h}_1 + \theta_1 = \omega_1 + i\omega_2$ never vanishes, so $\mathfrak{h}_2 = \theta_2 = \mathfrak{h}_3 = \theta_3$. We deduce that A satisfies Chen's equality from (6)-(7) and Definition 8.1. \square

From Theorem 7.7 and Propositions 8.5 and 8.9 we have the following immediate corollary.

Corollary 8.10 *Use the notation of Definitions 7.2, 7.4, 8.4 and 8.8. Every isotropic minimal surface $\mathbf{u} : \Sigma \rightarrow \mathcal{S}^6$ defines a linear pseudoholomorphic curve $\mathbf{c} : \Sigma \rightarrow \mathcal{C}$ such that $\tau \circ \mathbf{c} = \mathbf{u}$. Moreover, every linear pseudoholomorphic curve in \mathcal{C} is the lift of an isotropic minimal surface in \mathcal{S}^6 .*

Note Combining Theorem 7.7, Proposition 8.9 and Corollary 8.10 we deduce Theorem 1.5.

We conclude this section with the following observations. The horizontal and vertical pseudoholomorphic curves in \mathcal{C} , as remarked in Definition 8.7, can be regarded as holomorphic curves and thus admit *Weierstrass representations*; i.e. they can be described using holomorphic data. In contrast, isotropic minimal surfaces in \mathcal{S}^6 include the pseudoholomorphic curves in \mathcal{S}^6 which do *not* admit a Weierstrass representation. Therefore linear pseudoholomorphic curves in \mathcal{C} , and thus associative submanifolds in \mathcal{S}^7 satisfying Chen's equality, cannot be described purely using holomorphic data.

We also observe that not every lift of an isotropic minimal surface in \mathcal{S}^6 is necessarily linear since the pseudoholomorphic lifts to \mathcal{C} of isotropic minimal surfaces depend on 8 functions of 1 variable locally, whereas the linear pseudoholomorphic curves only depend on 6 functions of 1 variable.

9 Isometric embeddings

In this section we consider the following natural question: if there exists an embedding ι of a Riemannian 3-manifold (A, g_A) in (\mathcal{S}^7, g) as an associative 3-fold such that $\iota^*(g) = g_A$, is ι necessarily unique up to rigid motion?

Necessary conditions for ι to exist are given by the Gauss, Codazzi and Ricci equations (22)-(24). Since these impose strong restrictions on g_A and the second fundamental form of ι , we would expect in general that ι would be unique up to rigid motion. However, we find that if the associative submanifold is *ruled*, then ι need not be unique. We therefore restrict our attention to this situation.

Recall that ruled associative 3-folds in \mathcal{S}^7 are in one-to-one correspondence with pseudoholomorphic curves in the space \mathcal{C} of oriented geodesic circles in \mathcal{S}^7 by Proposition 7.3. We gave the second structure equations for the adapted frame bundle of a pseudoholomorphic curve Γ in \mathcal{C} in Proposition 8.6 in terms of vectors of 1-forms $\mathfrak{h} = (\mathfrak{h}_1, \mathfrak{h}_2, \mathfrak{h}_3)^T$ and $\theta = (\theta_1, \theta_2, \theta_3)^T$, a skew-hermitian matrix of 1-forms $\kappa = (\kappa_{jk})$ and a real 1-form $\rho = i \operatorname{Tr} \kappa$. We see from (6)-(10) that the forms $\mathfrak{h}_1, \theta_1, \rho, \kappa_{11}, \kappa_{22}$ and κ_{33} depend on the Levi-Civita connection of the corresponding ruled associative 3-fold A in \mathcal{S}^7 . In contrast, κ_{32} depends on components of the normal connection of A which are not necessarily determined by the Levi-Civita connection of A , and $\mathfrak{h}_2, \theta_2, \mathfrak{h}_3, \theta_3, \kappa_{21}$ and κ_{31} are determined by the second fundamental form of A .

The structure equations (72)-(74) are necessary conditions for a pseudoholomorphic embedding satisfying (70)-(71) to exist. Thus, these equations may be interpreted as necessary conditions for a ruled associative embedding to exist satisfying the structure equations (17)-(19).

We may always adapt frames on a pseudoholomorphic curve so that $\mathfrak{h}_3 = \theta_3 = 0$. Observe that (72)-(74) are invariant under the following transformation:

$$(\mathfrak{h}_2, \theta_2, \kappa_{21}, \kappa_{31}) \mapsto e^{ia}(\mathfrak{h}_2, \theta_2, \kappa_{21}, \kappa_{31}) \quad (75)$$

for a real constant a . Moreover, if we adapt frames further so that $\kappa_{31} = 0$, which we are always free to do, then (72)-(74) are also invariant under

$$\kappa_{32} \mapsto e^{ib} \kappa_{32} \quad (76)$$

for some real constant b .

Thus, given a solution to (70)-(74), we can potentially construct a two-parameter family of solutions with the same choice of $\theta_1, \mathfrak{h}_1, \kappa_{11}, \kappa_{22}$ and κ_{33} . The parameter a in (75) always gives a non-trivial family of solutions unless $\mathfrak{h}_2 = \theta_2 = \kappa_{21} = \kappa_{31} = 0$, which corresponds to the associative 3-fold A being simple. The parameter b in (76) will also give a non-trivial family of solutions for non-zero κ_{32} unless κ_{32} depends on the Levi-Civita connection of A , which will occur, for example, if A is Lagrangian (where $\kappa_{32} = \theta_1$) or minimal Legendrian (where $\kappa_{32} = -\mathfrak{h}_1$).

We conclude that, given a ruled, non-simple, associative isometric embedding of (A, g_A) in \mathcal{S}^7 , we hope to have at least an \mathcal{S}^1 -family of pseudoholomorphic embeddings of a surface Γ in \mathcal{C} which define an \mathcal{S}^1 -family of ruled associative isometric embeddings of (A, g_A) . Moreover, the family of isometric associative 3-folds will be non-congruent if they are not homogeneous because the eigendirections of the second fundamental form are different by (6)-(10) and (75).

The big problem is that the parameter a is only guaranteed to be globally defined on the simply connected cover of the surface Γ . The question of when the parameter a is well-defined on surfaces of positive genus is very difficult to answer in general: when $\Gamma \cong T^2$ this is known as the “period problem”. However, we can of course overcome this difficulty by considering $\Gamma \cong \mathcal{S}^2$. In this case, the structure equations (72)-(74) become necessary and sufficient conditions for the existence of a pseudoholomorphic embedding of Γ , and thus of a ruled associative isometric embedding of (A, g_A) .

Using the well-developed theory of minimal 2-spheres in \mathcal{S}^6 we have the following theorem.

Theorem 9.1 *Every non-totally geodesic minimal \mathcal{S}^2 in \mathcal{S}^6 has a horizontal pseudoholomorphic lift to \mathcal{C} , in the sense of Definition 8.7.*

Proof: Let Σ be a non-totally geodesic minimal 2-sphere in \mathcal{S}^6 . By the work in [10], Σ is linearly full in a totally geodesic \mathcal{S}^{2m} for $m = 2$ or 3 and Σ is superminimal, hence isotropic in the sense of Definition 8.4. An idea due to Chern [13] is to associate to Σ its so-called directrix curve Ξ , which is a holomorphic curve in \mathbb{CP}^{2m} . In fact, Ξ is a totally isotropic curve in \mathbb{CP}^{2m} , i.e. if $\xi : \Xi \rightarrow \mathbb{CP}^{2m}$ then $(\xi, \xi) = (\xi', \xi') = \dots = (\xi^{m-1}, \xi^{m-1}) = 0$, where $(,)$ denotes the complex inner product on \mathbb{CP}^{2m} . Thus Ξ defines a curve Γ in the space H_m of totally isotropic m -dimensional subspaces of \mathbb{C}^{2m+1} by sending each point in Ξ to the subspace spanned by ξ and its first $(m-1)$ derivatives at that point. Clearly, H_m is contained in a complex Grassmannian and thus in a complex projective space. It follows from the work in [2] that $H_m \cong \mathrm{SO}(2m+1)/\mathrm{U}(m)$ and that Γ is a holomorphic curve in H_m which is horizontal with respect to the fibration of $\mathrm{SO}(2m+1)/\mathrm{U}(m)$ over \mathcal{S}^{2m} .

Thus there is a well-defined horizontal lift, which we also call Γ , of Σ to $\mathcal{C} \cong \mathrm{Spin}(7)/\mathrm{U}(3)$ which is *holomorphic* with respect to the integrable complex structure $I_{\mathcal{C}}$ on \mathcal{C} introduced in Definition 7.2. However, by [19, Lemma 5.2 & Corollary 5.3], the complex structures $J_{\mathcal{C}}$ and $I_{\mathcal{C}}$ on \mathcal{C} agree on the horizontal distribution \mathcal{H} given in Definition 8.7, and so Γ is pseudoholomorphic with respect to $J_{\mathcal{C}}$. \square

We can think of the horizontal pseudoholomorphic lift to \mathcal{C} of a minimal 2-sphere in \mathcal{S}^6 as a “twistor lift”.

Remark By the work in [2], [10], [11] and [17], the proof of Theorem 9.1 can be generalised to show that any *superminimal* surface in \mathcal{S}^6 has a horizontal pseudoholomorphic lift to \mathcal{C} .

From Theorem 9.1 and the discussion proceeding it we have our main result.

Theorem 9.2 *Let $\mathbf{u} : \mathcal{S}^2 \rightarrow \mathcal{S}^6$ be non-totally geodesic and minimal, and recall Example 7.6 and Definition 8.7. Let $\mathbf{c}_\Gamma : \mathcal{S}^2 \rightarrow \mathcal{C}$ be a horizontal pseudoholomorphic lift defined by a holomorphic curve Γ in $\mathcal{X}(\mathcal{S}^2)$ and let $A = A(\mathcal{S}^2, \mathbf{u}, \Gamma)$.*

There is an \mathcal{S}^1 -family of isometric associative submanifolds of \mathcal{S}^7 , containing A , which satisfy Chen’s equality. Moreover, the family consists of non-congruent associative submanifolds if $\mathbf{u} : \mathcal{S}^2 \rightarrow \mathcal{S}^6$ does not have constant curvature.

Note From Theorem 9.2 we immediately deduce Theorem 1.6.

By [2, Corollary 4.14], the area of a minimal 2-sphere Σ in \mathcal{S}^6 is always an integer multiple of 4π . Thus, we can define the *degree* d_Σ of Σ by $\text{Area}(\Sigma) = 4\pi d_\Sigma$. This is the same as the degree of the associated twistor lift: the horizontal pseudoholomorphic curve in \mathcal{C} viewed as a holomorphic curve in a complex projective space as in [2]. As already observed, by the work of Calabi [10], a minimal 2-sphere in \mathcal{S}^n must be linearly full in an even-dimensional totally geodesic hypersphere. The main results of [18] and [32] show that the moduli space of minimal 2-spheres of degree d which are full in \mathcal{S}^{2m} for $m = 2$ or 3 has dimension $2d + m^2$. Moreover, by [2, Theorem 6.19], every integer $d \geq 6$ is the degree of some minimal 2-sphere in \mathcal{S}^6 , so there is an arbitrarily large moduli space of minimal 2-spheres with the same area. These spheres will give rise to one-parameter families of isometric associative embeddings by Theorem 9.2.

Remarks One could attempt to use general results on pseudoholomorphic curves to find the expected dimension of the moduli space of pseudoholomorphic 2-spheres in \mathcal{C} . This will essentially reduce to a Chern class calculation which should be relatively easy given the structure equations in Proposition 8.6.

We now study Theorem 9.2 in some special cases.

Corollary 9.3 *Given any non-constant curvature minimal $\mathbf{u} : \mathcal{S}^2 \rightarrow \mathcal{S}^4$, there exists an \mathcal{S}^1 -family of non-congruent isometric minimal Legendrian submanifolds in \mathcal{S}^7 satisfying Chen’s inequality which are ruled over \mathcal{S}^2 .*

Proof: By Proposition 7.9(b) and its proof, we see that a horizontal lift of a minimal \mathcal{S}^2 in $\mathcal{S}^4 \subseteq \mathcal{S}^6$ will define a minimal Legendrian submanifold A of \mathcal{S}^7 ruled over \mathcal{S}^2 . By Theorem 9.2 we have an \mathcal{S}^1 -family of isometric associative 3-folds containing A . However, since the transformation (75) leaves the conditions given in Example 4.9 invariant, this family will consist of minimal Legendrian submanifolds. \square

In a similar manner, using Example 4.8, Proposition 7.9(c) and Theorem 9.2 we have the following result.

Corollary 9.4 *Given any non-constant curvature pseudoholomorphic $\mathbf{u} : \mathcal{S}^2 \rightarrow \mathcal{S}^6$, there exists an \mathcal{S}^1 -family of non-congruent isometric Lagrangian submanifolds in \mathcal{S}^6 satisfying Chen's equality which are ruled over \mathcal{S}^2 .*

Remark One can see this result directly from the structure equations for Lagrangians satisfying Chen's equality given in [34, §6.4]. The Lagrangians are tubes of radius $\frac{\pi}{2}$ in the second normal bundle of the pseudoholomorphic \mathcal{S}^2 .

As a final comment, it would be interesting to know whether a minimal 2-sphere could be lifted to a pseudoholomorphic curve with $\kappa_{32} \neq 0$ independent of the Levi-Civita connection. Necessarily such a 2-sphere would have to be linearly full in \mathcal{S}^6 but not pseudoholomorphic. Given this minimal \mathcal{S}^2 and its pseudoholomorphic lift, we would then be able to define a 2-torus family of isometric associative embeddings using (75)-(76).

Acknowledgements The author is indebted to Robert Bryant for enlightening discussions, insight and helpful advice. He would also like to thank Daniel Fox and Mark Haskins for useful conversations and comments.

The author is supported by an EPSRC Career Acceleration Fellowship.

References

- [1] C. Bär, *Real Killing Spinors and Holonomy*, Commun. Math. Phys. **154** (1993), 509–521.
- [2] J. L. M. Barbosa, *On Minimal Immersions of \mathcal{S}^2 in \mathcal{S}^{2m}* , Trans. Amer. Math. Soc. **210** (1975), 75–106.
- [3] D. E. Blair, F. Dillen, L. Verstraelen and L. Vrancken, *Calabi Curves as Holomorphic Legendre Curves and Chen's Inequality*, Kyungpook Math. J. **35** (1996), 407–416.

- [4] J. Bolton, L. Vrancken and L. M. Woodward, *On Almost Complex Curves in the Nearly Kähler 6-Sphere*, Q. J. Math. **45** (1994), 407–427.
- [5] O. Borůvka, *Sur les Surfaces Représentées par les Fonctions Sphériques de Première Espèce*, J. Math. Pures et Appl. **12** (1933), 337–383.
- [6] R. L. Bryant, *Submanifolds and Special Structures on the Octonians*, J. Differential Geom. **17** (1982), 185–232.
- [7] R. L. Bryant, *Conformal and Minimal Immersions of Compact Surfaces into the 4-Sphere*, J. Diff. Geom. **17** (1982), 455–473.
- [8] R. L. Bryant, *Metrics with Exceptional Holonomy*, Ann. Math. **126** (1987), 525–576.
- [9] R. L. Bryant, *Some Remarks on G_2 -Structures*, in Proceedings of Gökova Geometry-Topology Conference 2005, pp. 75–109, edited by S. Akbulut, T. Önder and R. J. Stern, International Press, 2006.
- [10] E. Calabi, *Minimal Immersions of Surfaces in Euclidean Spheres*, J. Diff. Geom. **1** (1967), 111–125.
- [11] E. Calabi, *Quelques Applications de l'Analyse Complexe aux Surfaces d'Aire Minima*, in Topics in Complex Manifolds, pp. 59–81, edited by H. Rossi, University of Montreal, 1968.
- [12] B. Y. Chen, *Some Pinching and Classification Theorems for Minimal Submanifolds*, Arch. Math. (Basel) **60** (1993), 568–578.
- [13] S. S. Chern, *On the Minimal Immersions of the Two Sphere in a Space of Constant Curvature*, in Problems in Analysis: A Symposium in Honor of Salomon Bochner, pp. 27–40, edited by R. C. Gunning, Princeton University Press, Princeton, NJ, 1970.
- [14] F. Dillen, B. Opozda, L. Verstraelen and L. Vrancken, *On Totally Real 3-Dimensional Submanifolds of the Nearly Kaehler 6-Sphere*, Proc. Amer. Math. Soc. **99** (1987), 741–749.
- [15] F. Dillen and L. Vrancken, *Totally Real Submanifolds in $S^6(1)$ Satisfying Chen's Equality*, Trans. Amer. Math. Soc. **348** (1996), 1633–1646.
- [16] N. Ejiri, *Totally Real Submanifolds in a 6-Sphere*, Proc. Amer. Math. Soc. **83** (1981), 759–763.

- [17] L. Fernández, *On the Moduli Space of Superminimal Surfaces in Spheres*, Int. J. Math. Math. Sci. **44** (2003), 2803–2827.
- [18] L. Fernández, *The Dimension of the Space of Harmonic 2-Spheres in the 6-Sphere*, Bull. London Math. Soc. **38** (2006), 156–162.
- [19] D. Fox, *Cayley Cones Ruled by 2-Planes: Desingularization and Implications of the Twistor Fibration*, Comm. Anal. Geom. **16** (2008), 937–968.
- [20] Th. Friedrich, I. Kath, A. Moroianu and U. Semmelmann, *On Nearly Parallel G_2 -Structures*, J. Geom. Phys. **23** (1997), 259–286.
- [21] J. P. Gauntlett, D. Martelli, J. Sparks, D. Waldram, *A New Infinite Class of Sasaki–Einstein Manifolds*, Adv. Theor. Math. Phys. **8** (2004), 987–1000.
- [22] A. Gray, *Vector Cross Products on Manifolds*, Trans. Amer. Math. Soc. **141** (1969), 465–504.
- [23] A. Gray and P. S. Green, *Sphere Transitive Structures and the Triality Automorphism*, Pacific J. Math. **34** (1970), 83–96.
- [24] W. Gu and C. Pries, *Examples of Cayley Manifolds in \mathbb{R}^8* , Houston J. Math. **30** (2004), 55–87.
- [25] R. Harvey and H. B. Lawson, *Calibrated Geometries*, Acta Math. **148** (1982), 47–152.
- [26] M. Haskins, *Special Lagrangian Cones*, Amer. J. Math. **126** (2004), 845–871.
- [27] M. Haskins and N. Kapouleas, *Gluing Constructions of Special Lagrangian Cones*, in Handbook of Geometric Analysis, No. 1, pp. 77–145, Adv. Lect. Math. (ALM) **7**, Int. Press, Somerville, MA, 2008.
- [28] M. Haskins and N. Kapouleas, *Twisted Products and $SO(p) \times SO(q)$ -invariant Special Lagrangian Cones*, arXiv:1005.1419.
- [29] D. D. Joyce, *Riemannian Holonomy Groups and Calibrated Geometry*, Oxford Graduate Texts in Mathematics **12**, OUP, Oxford, 2007.
- [30] P.-F. Leung, *On the Curvature of Minimal Submanifolds in a Sphere*, Geom. Dedicata **56** (1995), 5–6.
- [31] A.-M. Li and J. Li, *An Intrinsic Rigidity Theorem for Minimal Submanifolds in a Sphere*, Arch. Math. **58** (1992), 582–594.

- [32] B. Loo, *The Space of Harmonic Maps of \mathcal{S}^2 into \mathcal{S}^4* , Trans. Amer. Math. Soc. **313** (1989), 81–102.
- [33] J. D. Lotay, *Calibrated Submanifolds of \mathbb{R}^7 and \mathbb{R}^8 with Symmetries*, Q. J. Math. **58** (2007), 53–70.
- [34] J. D. Lotay, *Ruled Lagrangian Submanifolds of the 6-Sphere*, to appear in Trans. Amer. Math. Soc., arXiv:0807.2084.
- [35] S. P. Marshall, *Some Special Lagrangian Submanifolds of \mathbb{C}^m* , dissertation, University of Oxford, 1999.
- [36] K. Mashimo, *Homogeneous Totally Real Submanifolds of \mathcal{S}^6* , Tsukuba J. Math. **9** (1985), 185–202.
- [37] B. O’Neill, *Isotropic and Kähler Immersions*, Canad. J. Math. **17** (1965), 907–915.
- [38] S. M. Salamon, *Harmonic and Holomorphic Maps*, in Geometry Seminar ‘Luigi Bianchi’ II - 1984, pp. 161–224, Lecture Notes in Mathematics **1164**, Springer Berlin, Heidelberg, 1985.
- [39] S. M. Salamon, *Riemannian Geometry and Holonomy Groups*, Pitman Research Notes in Mathematics **201**, Longman, Harlow, 1989.